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Membership separability: A new axiomatization of the Shapley value ☆

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ABSTRACT

The paper shows that Shapley's axiomatic characterization of his value can be strengthened considerably. Indeed, his additivity axiom can be replaced by a simple accounting property whereby a player's payoff is the difference of a reward based on the worth of coalitions to which she belongs, and a tax based on the worth of coalition to which she does not belong, without placing any restriction whatsoever on the functional relationship between the reward or the tax and the worths that determine them.

1. Introduction

The paper sheds new light on the Shapley value, one of the most successful solution concepts in cooperative game theory, and in particular how Shapley's (1953) original axiomatic result can be strengthened considerably.

His characterization result rests on three axioms. Both his anonymity and carrier axioms admit strong normative and positive interpretations. If two players play a same role in creating the surplus, then they will and should receive the same payoff (anonymity). If a group of players do not contribute to creating any surplus, then they will not or should not receive any payoff (carrier). Some have proposed justifications of the additivity axiom,[1] but it often comes across as a more technical requirement, in which case Shapley's result essentially amounts to investigating the carrier and anonymity axioms within the simpler class of linear[2] solutions.

We know that additivity does play a key role in Shapley's result. Schmeidler's (1969) nucleolus provides an example of a non-linear value that satisfies both the anonymity and carrier axioms. This note shows, however, that carrier and anonymity do characterize the Shapley value within a remarkably large class of non-linear values. In other words, Shapley's axiomatic result survives if one replaces his additivity axiom by a weaker axiom that does not rule out as many values. Shapley's core ideas are thus mathematically more robust than originally thought, as the anonymity and carrier axioms alone appear harder to meet than originally thought.

The new axiom that replaces additivity can be thought of as a simple accounting method. When considering a player, I propose to consider separately a reward she will receive based on the worth of coalitions to which she belongs, and a tax she will pay based on the worth of coalitions to which she does not belong. The axiom requires that her payoff be the

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1 Shapley thinks for instance of his value as determining an expected payoff in a game, and suggests additivity as a property of expected utility.

2 As is well-known, additivity is not exactly the same as linearity, but the distinction between the two properties is not relevant in the context of our discussion.

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reward minus the tax, without placing any restriction at all on the functional relationship between the reward or the tax and the worths that determine them.

There have been other attempts in the past to dispense with the additivity axiom. Young's (1985) marginality axiom postulates that a player's payoff depends only on her marginal contributions to the different coalitions to which she belongs. Thus it allows the value to be any function applied to inputs that are derived as differences of coalitional worths. By contrast, the axiom proposed here encompasses a different class of values where coalitional worths are not manipulated, but the impact of coalitional worths on the final payments can be separated based on membership. I will also show that the new axiom is weaker than Feltkamp's (1995) transfer axiom (first introduced by Dubey, 1975, on the class of simple games). In addition to a few more axiomatic results in that vein, there are also other approaches to characterize the Shapley value without relying on a property of additivity, using for instance ideas of balanced contributions, reduced game properties, or following the Nash program. The interested reader is referred to Myerson (1980), Hart and Mas-Colell (1989), Chun (1989), van den Brink (2001), Hamiache (2001), Kamijo and Kongo (2010), and Casajus (2011, 2014). Finally, Eisenman (1967) and Béal et al. (2012) show how the Shapley value can be computed as some average of 'compensations' where members of a coalition receive an equal share of its worth, and pay an equal share of the complement's worth. They thus provide instances where the Shapley value is presented explicitly via a tax-reward structure.

2. Reminder

The set of players is denoted by N. *Coalitions* are nonempty subsets of N. The set of all coalitions is denoted by $P(N)$. A characteristic function associates to every coalition a real number that represents the amount to be shared by its member should they cooperate. A *value* associates a payoff vector in $\mathbb{R}^N$ to every characteristic function. The Shapley value for instance is a weighted sum of the players's marginal contributions:

$$Sh_i(v) = \sum_{S \in P(N) | i \in S} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S \setminus \{i\})),$$

for each player i and each characteristic function v, where $v(\emptyset) = 0$, $s = \#S$, $n = \#N$.

Let v be a characteristic function and let x a payoff vector in $\mathbb{R}^N$. If π is a permutation of N, then $\pi(v)$ is the characteristic function defined as follows:

$$\pi(v)(S) = v(\pi(S)),$$

for each coalition S. Similarly, the vector $\pi(x)$ is defined as follows:

$$\pi(x)_i = x_{\pi(i)},$$

for each player i. A coalition S is said to be a *carrier* for a characteristic function v if $v(T) = v(S \cap T)$, for all coalitions T.

Shapley introduced the following axioms for a value σ.[3]

Anonymity (AN) *If π is a permutation of N, then $\sigma(\pi(v)) = \pi(\sigma(v))$, for each characteristic function v.*

Carrier (C) *If S is a carrier, then $\sum_{i \in S} \sigma_i(v) = v(S)$.*

3. Main result

I start by stating a new axiom.

Difference Formula (DF) *For each $i \in N$, there exist a function*

$$r_i : \mathbb{R}^{\{S \in P(N) | i \in S\}} \to \mathbb{R}$$

and a function

$$t_i : \mathbb{R}^{\{S \in P(N) | i \notin S\}} \to \mathbb{R}$$

such that

$$\sigma_i(v) = r_i([v(S)]_{S | i \in S}) - t_i([v(S)]_{S | i \notin S}),$$

for each characteristic function v.[4]

[3] The carrier axiom is equivalent to the combination of the axioms of "efficiency" and "null player" found sometimes in presentations of Shapley's result.

[4] One could require the functions r_i and t_i to take only nonnegative values, so as to interpretation of a reward and a tax. The characterization result holds even without requiring these functions to be positive.

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