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Indexing gamble desirability by extending proportional stochastic dominance

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ABSTRACT

We characterise two new orders of desirability of gambles (risky assets) that are natural extensions of the stochastic dominance order to complete orders, based on choosing optimal proportions of gambles. These orders are represented by indices, which we term the S index and the G index, that are characterised axiomatically and by wealth and utility uniform dominance concepts. The S index can be viewed as a generalised Sharpe ratio, and the G index can be used for maximising the growth path of a portfolio.

1. Introduction

1.1. Risk measurement and gamble desirability

Given a choice between two gambles (or risky assets, or securities), which one is more desirable? On which should one put one's money, and how much? The answer to such questions is often: it depends. That is, the answer is subjective and depends on the utility function for money of the agent charged with choosing.

There are many cases, however, in which a subjective answer is unsatisfactory. A pension management firm that pools funds from a broad range of clients, for example, needs in a sense to consider itself acting as a representative agent with an 'objective' capacity to sort potential gambles by desirability. But how is an objective ranking obtained?

In recent years, two numerical objective measures of riskiness introduced into the literature have garnered much attention: the Aumann–Serrano index (Aumann and Serrano, 2008) and the Foster–Hart measure (Foster and Hart, 2009). Both of these are based on the paradigm of acceptance or rejection of gambles; in other words agents are asked with respect to each gamble g whether they are willing to accept the terms of the gamble or prefer to avoid it. Gambles g and h are then ranked, in a general sense, by asking whether gamble g is accepted more, and thus rejected less, than h.[1]

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[1] The similarities and the differences between these two indices have been studied in several papers, including Foster and Hart (2013), Hart (2011), and Kadan and Liu (2014). Schreiber (2014) shows that the two indices coincide in the continuous time setup.

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1.2. The optimal proportions approach

In many realistic situations, however, a 'take it or leave it' approach to risky assets is not the norm; investors may instead select a *proportion* of an offered asset, with an attendant scaling of both positive and negative payoffs. This significantly shifts the perspective on the matter. For example, instead of asking about the certainty equivalent of a gamble g of an agent with utility u and wealth w, one inquires about the certainty equivalent of the *optimal proportion* of g taken by an agent with utility u at wealth w.

In effect, an agent is now not being asked to compare a gamble g directly with another gamble h. Instead, the focus changes to two *families* of gambles, namely positive scalar multiples of g versus positive scalar multiples of h, and the agent is in effect asked to select his or her optimal gamble in each of these two families and to compare *those* optimal gambles to each other.

Starting from this observation, we study here the topic of ranking gambles when agents may choose optimal proportions of gambles offered to them. Many of the same issues that are involved in comparing gambles in the non-proportional case arise in the proportional setting, namely finding an ordering that is both objective and *complete*, in the precise sense of enabling comparison of any pair of gambles.

Completeness is conspicuously lacking in several broadly used indices of gamble desirability. The Sharpe ratio, for example, fails to extend as an index for gambles that are not normally distributed. Similarly, the most widely accepted orderings on gambles, the n-th degree stochastic dominance orders (Hadar and Russell, 1969; Hanoch and Levy, 1969), are objective but none of them are total orderings, in any degree. Stochastic dominance was long ago extended to the proportional gamble setting (see Levy, 2006), but this extended ordering is, again, a partial ordering for comparing gambles.

In Section 5, we propose two complete gamble desirability indices, denoted S and G, that are of homogeneity zero and extend stochastic dominance. The motivation is a list of axiomatic desiderata that one would reasonably want such desirability indices to satisfy. The S index, when suitably extended to continuum state spaces, extends the Sharpe ratio; the G index ranks gambles according to their optimal growth path.

1.3. Duality

Our two indices, S and G, are by their definitions naturally 'dual' to each other, with S a CARA-based index and G a CRRA-based index, but they are also dual in another, more interesting way. There are many possible complete indices on gambles that extend stochastic dominance, but S and G do so 'uniformly', as we now explain. Stochastic dominance, by design, ranks a gamble g higher than h if every agent with a utility function located within a specified class of utilities prefers g to h. One may regard this as almost defining a voting mechanism: g ranks higher than h if and only if every agent in a specified class 'votes' for g over h in preference.

Unfortunately, the broader the class of participating agents, the more difficult it is to attain unanimity. Denote by $CE^*(u, w, g)$ the certainty equivalence an agent with utility u and wealth w ascribes to his or her optimal proportion of g. Then if one were to try to rank gamble g over h by asking each agent to point to the optimal proportions of g and h and then checking whether $CE^*(u, w, g) > CE^*(u, w, h)$, uniformity of voting preferences will almost certainly not be attained; even worse, a particular agent may rank g above h and then h above g, depending on the wealth w.

To remedy this, in Section 6.1 we rank g and h under a much more restrictive condition. Suppose that an agent with utility u states that he considers a gamble g worthy of consideration if and only if it meets some minimum criterion, for example if and only if $CE^*(u, w, g) > c$, for some fixed c. The gamble g is then regarded as uniformly undesirable by the agent if it *never* meets that criterion, at any wealth, i.e., $CE^*(u, w, g) \leq c$ for all w.

Given a pair of gambles g and h we can then ask: is it the case that for any agent, with any utility u, if g is uniformly undesirable then h must also be uniformly undesirable? If so, then we say that g wealth uniformly dominates h. We thus define a new order on gambles. It is immediately clear that it is a partial order, but somewhat surprisingly it turns out that wealth uniform dominance is not only a total ordering, it defines the same order as the S index and hence monotonically extends stochastic dominance to all orders.

This brings us back to the Aumann–Serrano index and the Foster–Hart measure. The Aumann–Serrano index is based on CARA utilities and the Foster–Hart measure on logarithmic utilities. Both indices, in the standard paradigm of acceptance or rejection of gambles, define total orders on gambles that extend the stochastic dominance order. Moreover, Hart (2011) shows that the two orders are related to each other in a 'dual' type of relation, with Aumann–Serrano following from 'wealth uniformity' and Foster–Hart from 'utility uniformity'.

The S-index of this paper bears a similarity to the Aumann–Serrano index. Both indices are related to CARA utility functions, and both are related to concepts involving 'wealth uniformity'. This then leads to the following question: if the S index is the proportional gambles parallel to the Aumann–Serrano index, via a wealth uniformity concept, what is the index that parallels the Foster–Hart measure, via an appropriately defined utility uniformity concept? In Section 6.3, we define such a utility uniformity concept within our framework and show that it does indeed lead to a complete order extending stochastic dominance – which, it turns out, is representable by our G index, completing the parallelism.