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The stable fixtures problem with payments ☆

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ABSTRACT

We consider multiple partners matching games (G, b, w), where G is a graph with an integer vertex capacity function b and an edge weighting w. If G is bipartite, these games are called multiple partners assignment games. We give a polynomial-time algorithm that either finds that a given multiple partners matching game has no stable solution, or obtains a stable solution. We characterize the set of stable solutions of a multiple partners matching game in two different ways and show how this leads to simple proofs for a number of results of Sotomayor (1992, 1999, 2007) for multiple partners assignment games and to generalizations of some of these results to multiple partners matching games. We also perform a study on the core of multiple partners matching games. We prove that the problem of deciding if an allocation belongs to the core jumps from being polynomial-time solvable for $b \le 2$ to NP-complete for $b \equiv 3$.

1. Introduction

Consider a group of soccer teams participating in a series of friendly games with each other off-season. Suppose each team has some specific target number of games it wants to play. For logistic reasons, not every two teams can play against each other. Each game brings in some revenue, which is to be shared by the two teams involved. The revenue of a game may depend on several factors, such as the popularity of the two teams involved or the soccer stadium in which the game is played. In particular, at the time when the schedule for these games is prepared, the expected gain may well depend on future outcomes in the current season (which are in general difficult to predict (Kern and Paulusma, 2001)). In this paper, we assume for simplicity that the revenues are known. Is it possible to construct a *stable* fixture of games, that is, a schedule such that there exist no two unmatched teams that are both better off by playing against each other? Note that if teams decide to play against each other, they may first need to cancel one of their other games in order not to exceed their targets.

The above example describes the problem introduced in this paper (see Section 4 for another example). In the next section we explain how we model this problem.

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1.1. Our model

We model the above example in two settings, namely as a matching problem and as a cooperative game. As we will show these two settings are deeply interwoven.

Matching problem. A *multiple partners matching game* is a triple (G, b, w), where $G = (N, E)$ is a finite undirected graph on n vertices and m edges with no self-loops and no multiple edges, $b : N \to \mathbb{Z}_+$ is a *vertex capacity function*, which is a nonnegative integer function, and $w : E \to \mathbb{R}_+$ is a nonnegative edge weighting. The set N is called the *player set*. There exists an edge $ij \in E$ if and only if players i, j can form a 2-player coalition. A set $M \subseteq E$ is a *b-matching* if every player i is incident to at most $b(i)$ edges of M. So, a b-matching is a set of 2-player coalitions, in which no player is involved in more 2-player coalitions than described by her capacity. If $ij \in M$ then i and j are *matched* by M; we also say that i and j are *partners* under M. The *value* of a 2-player coalition i, j with $ij \in E$ is given by $w(ij)$.

A nonnegative function $p : N \times N \to \mathbb{R}_+$ is a *payoff vector* with respect to a b-matching M if the following two conditions hold:

- $p(i, j) + p(j, i) = w(ij)$ for all $ij \in M$;
- $p(i, j) = p(j, i) = 0$ for all $ij \notin M$.

Here, $p(i, j)$ and $p(j, i)$ represent payoffs that i and respectively j obtain when they are matched to each other. If the two conditions above hold, then we say that M and p are *compatible*.[3] Note that p prescribes how the value $w(ij)$ of a 2-player coalition $\{i, j\}$ is distributed amongst i and j, ensuring that non-coalitions between two players yield a zero payoff. A pair (M, p), where M is a b-matching and p is a payoff compatible with M, is a *solution* for (G, b, w). We view p as a vector with entries $p(i, j)$, which we call *payoffs*.

Let (M, p) be a solution. Two players i, j with $ij \in E \setminus M$ may decide to form a new 2-player coalition if they are "better off", even if one or both of them must first leave an existing 2-player coalition in M (in order not to exceed their individual capacity). To describe this formally we define a *utility function* $u_p : N \to \mathbb{R}_+$, related to a payoff vector p. If i is *saturated* by M, that is, if i is incident with $b(i)$ edges in M, then we let $u_p(i) = \min\{p(i, j) : ij \in M\}$ be the worst payoff $p(i, j)$ of any 2-player coalition i is involved in. Otherwise, i is *unsaturated* by M and we define $u_p(i) = 0$. Alternatively, we could define $u_p(i)$ as the $b(i)$th largest payoff $p(i, j)$ to i. Note that the second definition shows that utilities are independent of M and determined by p only (recall[1] that this is because p in fact determines M).

A pair i, j with $ij \in E \setminus M$ *blocks* (M, p) if $u_p(i) + u_p(j) < w(ij)$. We say that (M, p) is *stable* if it has no blocking pairs, or equivalently, if every edge $ij \in E \setminus M$ satisfies the *stability condition*, that is, if

$$u_p(i) + u_p(j) \geq w(ij) \text{ for all } ij \in E \setminus M.$$

Note that the stability condition only needs to be verified for edges not in M.

Remark 1. Let (G, b, w) be a multiple partners matching game with $b \equiv 1$. Then any b-matching is a 1-matching, i.e, a *matching*, as for each $i \in N$, we have $p(i, j) > 0$ for at most one player $j \neq i$, which must be matched to i. In that case we will sometimes assume, with slight abuse of notation, that p is a nonnegative function defined on N. Then we can write $u_p(i) = p(i)$ for every $i \in N$. Checking whether a pair (M, p) is a solution for $(G, 1, w)$ comes down to verifying whether $p(i) + p(j) = w(ij)$ holds for every edge $ij \in M$. Checking whether a solution (M, p) is stable comes down to verifying whether $p(i) + p(j) \geq w(ij)$ holds for every edge $ij \in E \setminus M$.

We can now define our problem formally:

> STABLE FIXTURES WITH PAYMENTS (SFP)
> *Instance:* a multiple partners matching game (G, b, w)
> *Question:* does (G, b, w) have a stable solution?

Example 1. Let G be the 4-vertex cycle $u_1v_1u_2v_2u_1$ displayed in Fig. 1. Let $b \equiv 1$ and $w \equiv 1$. Then G has two maximum weight matchings, namely $M = \{u_1v_1, u_2v_2\}$ and $\hat{M} = \{u_1v_2, u_2v_1\}$. Let p be given by $p(u_1, v_1) = \frac{7}{10}$, $p(v_1, u_1) = \frac{3}{10}$, $p(u_2, v_2) = \frac{7}{10}$, $p(v_2, u_2) = \frac{3}{10}$ and $p(u_1, v_2) = p(v_2, u_1) = p(u_2, v_1) = p(v_1, u_2) = 0$. Then p is compatible with M and (M, p) is a stable solution for $(G, 1, 1)$. We also observe that p is not compatible with $\hat{M}$. However, there exists a stable solution $(\hat{M}, \hat{p})$, where $\hat{p}$ can be obtained from p by permuting the entries $p(i, j)$ for every fixed i. Namely, let $\hat{p}$ be defined as $\hat{p}(u_1, v_2) = \frac{7}{10}$, $\hat{p}(v_2, u_1) = \frac{3}{10}$, $\hat{p}(u_2, v_1) = \frac{7}{10}$, $\hat{p}(v_1, u_2) = \frac{3}{10}$ and $\hat{p}(u_1, v_1) = \hat{p}(v_1, u_1) = \hat{p}(u_2, v_2) = \hat{p}(v_2, u_2) = 0$.

[3] Assume that b and w are strictly positive functions. If p and M are compatible, then p is not compatible with any other b-matching, that is, p uniquely determines M. However, for our purposes it is more convenient to follow the literature and define p with respect to M.

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