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Nonatomic potential games: the continuous strategy case ☆

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ABSTRACT

This paper studies large population (nonatomic) potential games with continuous strategy sets. We define such games as population games in which the payoff function is equal to the gradient of a real-valued function called the potential function. The Cournot competition model with continuous player set and continuous strategy set is our main example and is analyzed in detail. For general potential games, we establish that maximizers of potential functions are Nash equilibria. For a particular class of potential games called aggregative potential games, we characterize Nash equilibria using a one-dimensional analogue of the potential function, which we call the quasi-potential function. Finally, we show that a large population potential game is the limit of a sequence of finite-player potential games as the number of players approaches infinity.

1. Introduction

Among the many contributions of Lloyd Shapley to game theory, two of the most significant ones are the concepts on nonatomic games (Aumann and Shapley, 1974) and potential games (Monderer and Shapley, 1996a). In nonatomic games, no individual player has a significant effect on the outcome of the game. Such games are useful in modeling economic situations with a large number of very "small" agents. Potential games are games in which players' incentives to change strategies can be captured using a common real-valued function called the potential function. Monderer and Shapley (1996a) define potential games with a finite number of players. Finite-player potential games are of interest because, as Monderer and Shapley (1996a) show, the existence of a pure Nash equilibrium follows from maximizing the potential function. Furthermore, Monderer and Shapley (1996b) show that the well-known learning mechanism of fictitious play converges to Nash equilibria in such games.

Sandholm (2001) combines the two notions of nonatomic games and potential games to define potential games with finite strategy sets for large populations. In large population games, no individual player has an impact on the aggregate population behavior or on other players' payoffs. In such a context, Sandholm (2001) defines a potential game as a population game in which the payoff function is equal to the gradient of a real-valued function called the potential function. Sandholm (2001) establishes that this is an appropriate extension of Monderer and Shapley (1996a)'s definition of a potential game since a large population potential game defined this way is the limit of a convergent sequence of finite-player

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potential games as the number of players approaches infinity. One of the key results in that paper, which is analogous to the result in Monderer and Shapley (1996a), is that maximizers of the potential function are Nash equilibria of the potential game.

This paper contributes to the literature on large population (and hence nonatomic) potential games by developing a general theory of such games in the setting of continuous strategy sets. This exercise is of interest for two reasons. First, certain interesting economic applications of potential games are naturally modeled as continuous strategy models. A typical example is the Cournot competition model. Monderer and Shapley (1996a) identify a finite-player Cournot competition model with a continuous strategy set as a potential game. Sandholm (2010a) also establishes a similar result: the large population version of the Cournot competition model is a potential game as defined in Sandholm (2001).[1] However, that example is restricted to the finite strategy setting. The approach of the present paper allows us to present such large population potential games of economic significance, like the Cournot competition model, in the more natural situation of continuous strategy settings. In fact, as we elaborate further below, the Cournot competition model is an example of a special class of potential games with economic significance, which we call aggregative potential games.

Second, the analysis of nonatomic potential games with continuous strategy sets raises certain interesting issues of mathematical and technical nature. In any population game with a continuous strategy set, the state space is the set of probability measures over the strategy set. Therefore, describing a potential game in such a state space requires the resolution of such non-trivial problems like the appropriate choice of topology on the state space, the definition of the potential function and the characterization of Nash equilibria of such games in terms of the potential function. Such difficulties ensure that extending the concept of large population potential games to the setting of continuous strategy sets is a substantive and worthwhile exercise.

Some of the technical problems we have alluded to have already been considered in the existing literature on population games with continuous strategy sets. Oechssler and Riedel (2001, 2002) provide extensive discussions on the choice of appropriate topology in the space of probability measures. These papers, as well as Hofbauer et al. (2009), also examine a specific class of games called doubly symmetric games, which are examples of potential games. Cheung (2014, 2016) and Lahkar and Riedel (2015) provide a more general definition of potential games with continuous strategy sets. Analogous to the definition of a potential game in Sandholm (2001), they define a potential game as a population game in which the payoff function is equal to the gradient of a potential function. The key difference between this definition and the definition of finite-dimensional potential games in Sandholm (2001) is that the gradient needs to be defined in terms of Fréchet derivatives.

We adopt the same definition of potential games as in Cheung (2014, 2016) and Lahkar and Riedel (2015). The key contribution that this paper then makes to the general theory of potential games with continuous strategy sets is the characterization of Nash equilibria of such games. We show that a sufficient condition for a population state to be a Nash equilibrium of the potential game is that the population state is a local maximizer of the potential function. For the particular case where the potential function is concave on the state space, maximization of potential function is both necessary and sufficient for a population state to be a Nash equilibrium.

The potential function of a potential game with continuous strategy space is, however, defined over an abstract measure space. Hence, maximizing this function would in general be a highly complex exercise.[2] It would, therefore, be convenient to have certain elementary methods for maximizing the potential function and, thereby, identifying Nash equilibria of at least some forms of potential games. We provide such a technique for one class of potential games called aggregative potential games (Lahkar, 2017). These are games which, apart from possessing a potential function, also satisfy the definition of an aggregative game (Corchón, 1994) in the sense that payoffs depend upon the aggregate strategy level in the population. We consider aggregative potential games with negative externalities and extend the methodology used by Lahkar (2017) to analyze such games with finite strategy sets to the present setting of continuous strategy sets. We confine ourselves to negative externalities because equilibrium characterization is particularly elegant in this context. Further, the Cournot competition model, which is the main motivating example of this paper, is an aggregative potential game with negative externalities. Another significant economic example of this class of games is the tragedy of the commons. We note that our analysis of negative externalities in aggregative potential games also allows us to introduce and define the notion of externalities in general large population games with continuous strategy sets.

Another well-known example of potential games is the class of congestion games (Beckmann et al., 1956). As will be mentioned at the beginning of Section 2.1, to define our notion of potential game, we only need the strategy set to be a compact metric space. This allows us to define a large population congestion game with a finite set of routes, which is a natural assumption, but in which the intensity of usage of a route is a continuous variable. We show that this extended notion of a congestion game is a potential game.

We establish a formal connection of our notion of potential games with that of Monderer and Shapley (1996a). Following Sandholm (2001), we show that a nonatomic potential game is the limit of a sequence of finite-player potential games as the number of players goes to infinity, but in the continuous strategy setting. This result illustrates the generality of the

[1] Cf. Sandholm (2010a), Example 3.1.3.
[2] See, for example, Guignard (1969) for a general constrained maximization approach applicable to Banach spaces.

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