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The axiom of equivalence to individual power and the Banzhaf index

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ABSTRACT

I introduce a new axiom for power indices on the domain of finite simple games that requires the total power of any given pair i, j of players in any given game v to be equivalent to some individual power, i.e., equal to the power of *some* single player k in *some* game w. I show that the Banzhaf power index is uniquely characterized by this new "equivalence to individual power" axiom in conjunction with the standard semivalue axioms: transfer (which is the version of additivity adapted for simple games), symmetry or equal treatment, positivity (which is strengthened to avoid zeroing-out of the index on some games), and dummy.

1. Introduction

The Banzhaf power index (henceforth BI) is one of the best known methods for numerically representing the individual power of a voter in a voting situation by means of a simple computational formula. The basic idea behind BI was laid out already by Penrose (1946), but was later rediscovered and popularized by Banzhaf (1965, 1966, 1968).[1] Shapley (1977) and Dubey and Shapley (1979) initiated the migration of BI into the framework of cooperative game theory, and one of Dubey and Shapley's (1979) definitions of BI has since become standard. It is easy to describe – the power of a voter is defined as the probability that he is a "swinger", i.e., that his "yes" vote changes the outcome when all individuals cast their votes independently and with equal probability for "yes" and "no". Equivalently, as any voting situation is modeled as a simple cooperative game[2] with a finite player set N, the BI of player $i \in N$ is given by his probability to turn a random coalition of players from losing to winning by joining it, assuming that the coalition is chosen with respect to the uniform distribution over the subsets of $N \backslash \{i\}$.

Naturally, other probability distributions over the subsets of $N \backslash \{i\}$ can be considered, notably the one which leads to the famous Shapley–Shubik power index (henceforth SSI), introduced in Shapley and Shubik (1954). The distribution behind

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[1] For this reason BI is also known as the Penrose–Banzhaf index.

[2] The description of voting in terms of simple games was suggested in Shapley and Shubik (1954), and further elaborated upon in Shapley (1962).

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the SSI of player i combines two sequential random choices: first, the size of a coalition in $N\backslash\{i\}$ is determined, with all sizes being equally likely; second, a coalition is selected from the set of all (equally likely) coalitions with the previously determined size. A major distinctive feature of SSI is its *efficiency* – namely, the total power of all players is 1 in any simple game.[3] In contrast, BI is not efficient in general: the total power of all players in a simple game is equal to the expected number of "swingers" in N. The latter property has been elevated to the rank of an axiom by Dubey and Shapley (1979) as a substitute for efficiency, in an attempt to provide BI with an axiomatic foundation that would mirror that of SSI (established in Dubey, 1975).

The Dubey and Shapley axiom, however, may be deemed unsatisfactory,[4] not least because it explicitly relies on counting "swings" (the notion on which BI is based). Fortunately, BI has other distinguishing features that can replace this axiom, of which we shall mention just two. One is the *composition* property that was formally defined and proved by Owen (1975, 1978). It pertains to a two-tier voting process, and requires the power of player i in a compound voting game to be equal[5] to the product of i's power in the first-tier game in which he participates and the power of i's delegate in the second-tier game.[6] Another distinctive property of BI is *2-efficiency*. Established in Lehrer (1988), it requires the sum of the power of any two players, i and j, in any game v to be equal to the power of player i in the game $v_{i,j}$ obtained from v by "merging" j into i (i.e., any coalition that contains i in the game $v_{i,j}$ has the same worth as that coalition with the addition of j in the game v).

The 2-efficiency property is quite powerful. Lehrer (1988) showed that any 2-efficient power index that coincides with BI on the set of all 2-player simple games is, in fact, identical to BI on all games. But 2-efficiency is also powerful enough to be a basis for an axiomatization of BI that does not contain an explicit or implicit comparison to BI on certain games. Lehrer (1988) considered a weaker version of 2-efficiency, which he termed the *superadditivity* axiom, whereby the total power of any i, j in any v *does not exceed* the power of the merged player i in $v_{i,j}$. He proved that BI is uniquely characterized by the superadditivity axiom along with other requirements that are routinely imposed[7] on power indices (these are the *transfer*, *equal treatment* or *symmetry*, and *dummy* axioms).[8] Recently, Casajus (2012) showed that the symmetry axiom is not needed in Lehrer's characterization of BI (and that the three remaining axioms are logically independent). That is, superadditivity, transfer and dummy axioms uniquely characterize BI on the set of finite simple games.

In this work we introduce a new axiom, *equivalence to individual power (EIP)*, that is related to 2-efficiency but has an independent conceptual appeal. The EIP axiom is based on the idea that when trying to conceptualize the collective power of a pair of players, one need not leave the realm where only the individual power is defined, as the collective power has an ordinal equivalent in that realm. Formally, EIP postulates that, given any two players $i, j \in N$ and a simple game v on N, the total power of the pair i, j in v is equivalent to some individual power, i.e., equal to the power of *some* (single) player k in *some* simple game w. Only one mild assumption links w to the original game v: w should have the same – or smaller – carrier compared to v. There need not be any other relation between w and v. Thus, according to EIP, the "language" of individual power must be sufficiently "expressive" to also be able to capture the total power of pairs.[9] EIP hence suggests a conceptual simplification, or a shortcut, that an observer adept at comparing the power of various individual players across and within simple games can utilize when faced with the task of comparing the total power of pairs, as the latter may be reduced to an equivalent comparison of individual power. In mathematical terms, EIP requires the union of the image sets of all players' individual power indices to be sufficiently rich so as to contain the image sets of all 2-sums of individual power indices.

The usual formulations of the 2-efficiency property (such as those in Lehrer, 1988 and Casajus, 2012) treat the original simple game v and the merged $v_{i,j}$ as having different player sets (with j missing from the player set of the latter game). To allow comparison with EIP, we note that v and $v_{i,j}$ can be assumed to have the same set of players N, but different carriers: if v has a carrier $T \subset N$, then $T\backslash\{j\}$ acts as a carrier for $v_{i,j}$ (with j being a null player). With this convention, the axiomatization of BI in Lehrer (1988) holds with a fixed player set N.[10] The EIP axiom can thus be viewed as a weakening of the 2-efficiency property. Indeed, given a 2-efficient power index, a simple game v on N and $i, j \in N$, take w to be the merged game $v_{i,j}$ and k to be the merged player i; then any carrier of v is also a carrier of $v_{i,j} = w$, and, by 2-efficiency,

[3] In this paper we adopt the Dubey and Shapley (1979) notion of a simple game, which requires the grand coalition N to be winning and the game to be monotonic (i.e., a winning coalition must remain winning if it is expanded).

[4] See, e.g., Section 5 in Dubey et al. (2005).

[5] To be precise, a second-tier game needs to be decisive (namely, a winning coalition must have a losing complement in N, and vice versa) for the composition property to hold.

[6] A composition property-based axiomatization of the BI on the domain of simple games appeared in Dubey et al. (2005).

[7] See Dubey (1975), Dubey and Shapley (1979), Einy (1987).

[8] Superadditivity and 2-efficiency also figure prominently in axiomatizations of the Banzhaf *value* (BV), the extension of BI to the set of all games on N. Lehrer's (1988) Theorem B establishes a characterization of BV that is identical to that of BI, using the linearity axiom instead of transfer. See also the works of Nowak (1997) and Casajus (2011, 2012), where the linearity axiom is replaced by versions of Young's (1985) monotonicity.

[9] Were the statement of EIP to be made not just for pairs of players, but extended to require the total power of *any* number of players (or, at least, of triples) to be matchable by some individual power, BI would violate such an extension. Thus, our statement of EIP delineates the precise extent to which the total power is representable by individual power according to BI. See the discussion in Remark 2, where a softer, but less compelling version of EIP will be mentioned, that does allow for a rendering of the total power of multiple players in terms of the individual power.

[10] The axiomatization in Theorem 5 of Casajus (2012) that removes the symmetry axiom from Lehrer's list also holds for games with a fixed player set N, assuming $|N| \geq 3$.

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