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On a class of vertices of the core

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It is known that for supermodular TU-games, the vertices of the core are the marginal vectors, and this result remains true for games where the set of feasible coalitions is a distributive lattice. Such games are induced by a hierarchy (partial order) on players. We propose a larger class of vertices for games on distributive lattices, called min–max vertices, obtained by minimizing or maximizing in a given order the coordinates of a core element. We give a simple formula which does not need to solve an optimization problem to compute these vertices, valid for connected hierarchies and for the general case under some restrictions. We find under which conditions two different orders induce the same vertex for every game, and show that there exist balanced games whose core has vertices which are not min–max vertices if and only if $n > 4$.

1. Introduction

In the seminal paper of Shapley (1971) was undertaken probably the first study of the geometric properties of the core of TU-games. In particular, it was established that the set of marginal vectors coincides with the set of vertices of the core when the game is convex. This paper was the starting point of numerous publications on the core and its variants, studying its geometric structure (vertices, facets, since it is a closed convex polytope) for various classes of games (in particular, the assignment games of Shapley and Shubik, 1972).

In a parallel way, it was found that the classical view of TU-games, defined as set functions on the power set of the set of players, was too narrow, and the idea of restricted cooperation (i.e., not any coalition can form) germinated in several papers, most notably Aumann and Drèze (1974); Myerson (1977); Owen (1977) and Faigle (1989), who coined the term "restricted cooperation" and precisely studied the core of such games. Many algebraic structures were proposed for the set of feasible coalitions, e.g., (distributive) lattices, antimatroids, convex geometries, etc. It turned out that the structure of the core became much more complex to study, in particular due to the fact that the core on such games may become unbounded (see a survey in Grabisch, 2013). However, as shown by Derks and Gilles (1995), the main result established in Shapley (1971) remains true for games on distributive lattices: for supermodular games, the set of marginal vectors still coincides with the set of extreme points of the core.

The question addressed in this paper arises naturally from the last result: What if the game is not supermodular? Is it possible to know all of its vertices in an analytical form? The question has puzzled many researchers, and so far only partial answers have been obtained, and only in the case of classical TU-games, i.e., without restriction on cooperation. Significant contributions have been done in particular by Núñez and Rafels (1998), and Tijs (2005). In the former work, a family of

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vertices is obtained, which is shown to cover all vertices of the core when the game is almost convex (i.e., satisfying the supermodularity condition except when the grand coalition is involved). Later, Núñez and Rafels (2003) have shown that this family of vertices is also exhaustive for assignment games, while Trudeau and Vidal-Puga (2017) have shown that the same result holds for minimum cost spanning tree games. In the work of Tijs, another family of vertices is proposed, called leximals, which is leading to the concept of lexicore and the Alexia value.

The present paper lies in the continuity of these works, showing that the two previous families have close links, proposing a wider class of vertices (unfortunately, still not exhaustive in all cases), and most importantly, establishing results in the general context of games on distributive lattices. Such a class of games is of considerable interest, because it has a very simple interpretation: the set of feasible coalitions is induced by a hierarchy (partial order) on the set of players, and feasible coalitions correspond to subsets of players where every subordinate of a member must be present. In the absence of hierarchy, the classical case is recovered.

We summarize the main achievements of the paper. We first give a tight upper bound of the number of vertices of the core, using an argument of Derks and Kuipers (2002). Then we introduce the family of *min–max vertices*, obtained by minimizing or maximizing in a given order the coordinates of a core element. Minimization (respectively, maximization) is performed if the considered coordinate (player) is a minimal element (respectively, a maximal element) in the sub-hierarchy formed by the remaining players. We prove that these are indeed vertices of the core (Theorem 2), and that in the case of supermodular games, we recover all marginal vectors (Corollary 1). The case of connected hierarchies reveals to be particularly simple, because min–max vertices take a simple form and can be computed directly without solving an optimization problem (Theorem 5). In the general case, a similar computation can be done provided some conditions are satisfied (Formula (15)). Two different orders may yield the same min–max vertex for every game. We show in Theorem 7 that this arises if and only if one of the orders can be obtained from the other one by a sequence of switches exchanging minimal and maximal elements. Lastly, we investigate the limits of the min–max approach to find vertices, and show that there exist balanced games whose core has vertices which are not min–max vertices if and only if $n > 4$ (Theorem 8).

The paper is organized as follows. Section 2 introduces the necessary material for games on distributive lattices and cores of such games, which are unbounded in general. We show that the structure of the convex hull of the vertices of the core of games on distributive lattices is more complex than the structure of the core of ordinary games (Proposition 1). Section 3 gives an upper bound of the number of vertices of the core. Section 4 is the main section of the paper, introducing and studying min–max vertices. Section 5 investigates under which condition orders yield identical min–max vertices. Examples illustrating the main results and concepts are given in Section 6, together with a practical summary of how to proceed. The limits of the min–max approach are investigated in Section 7, and the paper finishes with Section 8 detailing the past literature on the topic.

2. Notation, definitions and preliminaries

A *partially ordered set* or *poset* $(P, \preceq)$ is a set P endowed with a partial order $\preceq$, i.e., a reflexive, antisymmetric and transitive binary relation. A poset $(P, \preceq)$ is a *lattice* if every two elements $x, y \in P$ have a supremum and an infimum, denoted respectively by $\vee, \wedge$. The lattice is distributive if $\vee, \wedge$ obey distributivity. As usual, $x \prec y$ means $x \preceq y$ and $x \neq y$. We say that x *covers* y, denoted by $y \prec\!\cdot\, x$, if $y \prec x$ and there is no $z \in P$ such that $y \prec z \prec x$. A *chain* in $(P, \preceq)$ is a sequence $x_0, \dots, x_p$ such that $x_0 \prec \cdots \prec x_p$, and its *length* is p. The *height* of $(P, \preceq)$ is the length of a longest chain in $(P, \preceq)$.

Throughout the paper we consider posets $(N, \preceq)$, where $N \subseteq U$ is finite, with $|N| = n$, and U is a set that contains $\{1, \dots, 5\}$. The set N can be considered as a set of players, agents, and $\preceq$ as expressing precedence constraints or hierarchical relations among players. For this reason, we will often refer to $(N, \preceq)$ as a hierarchy.

Subsets of N are called *coalitions*, and a coalition S is said to be *feasible* if $i \in S$ and $j \preceq i$ imply $j \in S$. In other words, the feasible coalitions are the downsets of the poset $(N, \preceq)$, and we denote by $\mathcal{O}(N, \preceq)$ the set of downsets of $(N, \preceq)$. It is well known that $(\mathcal{O}(N, \preceq), \subseteq)$ is a distributive lattice of height n, whose infimum and supremum are set intersection and union, respectively. By Birkhoff's (1933) Theorem, the converse also holds: any distributive lattice of height n is isomorphic to the set of downsets of some poset of n elements.

A (TU) *game with precedence constraints* (Faigle and Kern, 1992) is a triple $(N, \preceq, v)$ where $(N, \preceq)$ is a poset and $v : \mathcal{O}(N, \preceq) \to \mathbb{R}$ satisfies $v(\emptyset) = 0$. The set $\mathcal{O}(N, \preceq)$ of feasible coalitions is denoted by $\mathcal{F}$. Classical TU-games correspond to the case $\mathcal{F} = 2^N$, i.e., the partial order $\preceq$ is empty. We denote by Γ the set of games $(N, \preceq, v)$, $N \subseteq U$, with precedence constraints.

We say that $(N, \preceq)$ is *connected* if the Hasse diagram of $(N, \preceq)$, seen as a graph, is connected in the sense of graph theory, i.e., if for any two distinct $i, j \in N$, there is a sequence of elements $i = j_1, \dots, j_m = j$ in N such that either $j_\ell \prec\!\cdot\, j_{\ell+1}$ or $j_{\ell+1} \prec\!\cdot\, j_\ell$ for every $\ell = 1, \dots, m-1$. In this case, we speak of a *connected hierarchy*.

For any $x \in \mathbb{R}^N$, we use the shorthand $x(S) = \sum_{i \in S} x_i$ for any nonempty $S \in 2^N$. The *set of feasible payoff vectors* and the *set of preimputations* of a game $(N, \preceq, v)$ are respectively defined by

$$X^*(N, \preceq, v) = \{x \in \mathbb{R}^N : x(N) \leqslant v(N)\}, \quad X(N, \preceq, v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}.$$

The *core* of a game $(N, \preceq, v)$ is the set defined by

$$C(N, \preceq, v) = \{x \in X(N, \preceq, v) : x(S) \geqslant v(S), \forall S \in \mathcal{F}\}.$$

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