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Gross substitutability: An algorithmic survey

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ABSTRACT

The concept of gross substitute valuations was introduced by Kelso and Crawford as a sufficient conditions for the existence of Walrasian equilibria in economies with indivisible goods. The proof is algorithmic in nature: gross substitutes is exactly the condition that enables a natural price adjustment procedure – known as *Walrasian tatônnement* – to converge to equilibrium. The same concept was also introduced independently in other communities with different names: $M^\natural$-concave functions (Murota and Shioura), Matroidal and Well-Layered maps (Dress and Terhalle) and valuated matroids (Dress and Wenzel). Here we survey various definitions of gross substitutability and show their equivalence. We focus on algorithmic aspects of the various definitions. In particular, we highlight that gross substitutes are the exact class of valuations for which demand oracles can be computed via an ascending greedy algorithm. It also corresponds to a natural discrete analogue of concave functions: local maximizers correspond to global maximizers.

1. Gross substitutes and Walrasian tatônnement

The notion of *gross substitutes* was introduced by Kelso and Crawford (1982) in order to analyze two sided matching markets of workers and firms. Originally it was defined as a condition on the *gross product* generated by a set of workers for a given firm, hence the name *gross substitutes*. Such condition allowed a natural salary adjustment process to converge to a point where each worker is hired by some firm and no worker is over-demanded. Gul and Stacchetti (1999) later use the same notion to analyze the existence of price equilibria in markets with indivisible goods. For this survey, we adopt the Gul and Stacchetti terminology and talk about buyers/items/prices instead of firms/workers/salaries as in Kelso and Crawford.

Before we proceed, we fix some notation: we denote by $[n] = \{1, \dots, n\}$ a set of items (goods). A *valuation* over such items is a function $v : 2^{[n]} \to \mathbb{R}$ such that $v(\emptyset) = 0$.[1] Given a price vector $p \in \mathbb{R}^n$ and a set $S \subseteq [n]$, we denote $p(S) = \sum_{j \in S} p_j$. We will define $v_p(S) = v(S) - p(S)$ as the value of a subset S under the price vector p. This corresponds to the utility of an agent with this valuation for acquiring such set under those prices. Given disjoint sets S, T we define the *marginal value* of T with respect to S as $v(T|S) = v(T \cup S) - v(S)$. We sometimes omit braces in the representation of sets when this is clear from the context, for example, by $v(i, j|S)$ we denote $v(\{i, j\}|S)$ and by $S \cup j$ we denote $S \cup \{j\}$.

An *economy with indivisible goods* is composed by a set $[n]$ of items (goods) and $[m]$ of buyers (agents) where each agent $i \in [m]$ has a valuation $v^i : 2^{[n]} \to \mathbb{R}$. We use the notion of the *demand correspondence* to define an equilibrium of this economy:

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[1] Note that we don't require monotonicity in the definition. When we refer to a valuation for which $v(S) \le v(T)$ whenever $S \subseteq T$, we will refer to it as a *monotone* valuations or valuations satisfying *free-disposal*.

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Definition 1.1 *(Demand correspondence).* Given a valuation function $v : 2^{[n]} \to \mathbb{R}$ and a vector of prices $p \in \mathbb{R}^n$, we define the demand correspondence as the family of sets that maximize the utility of an agent under a price vector p:

$$D(v, p) := \{S \subseteq [n]; v_p(S) \geq v_p(T), \forall T \subseteq [n]\}$$

Definition 1.2 *(Walrasian equilibrium).* Given an economy with indivisible goods with n goods, m agents and valuations $\{v^i\}_i$ satisfying free-disposal,[2] a *Walrasian equilibrium* corresponds to a vector of prices $p \in \mathbb{R}^n_+$ and a partition of the goods in disjoint sets $[n] = \cup_{i=1}^m S_i$ such that $S_i \in D(v^i, p)$ for all i.

A reader familiar with the duality theorem in linear programming will readily recognize that the definition of Walrasian equilibrium closely resembles the *complementarity conditions* where the prices play the role of dual variables. Indeed, this is formalized by the results known as the First and Second Welfare Theorem. The First Welfare Theorem states that if $(p, S_1, \dots, S_m)$ is a Walrasian equilibrium then this partition corresponds to the optimal allocation of goods, i.e., the allocation maximizing $\sum_i v^i(S_i)$. The proof is quite elementary: let $S_1^*, \dots, S_m^*$ be any partition maximizing the welfare. Then since $S_i \in D(v^i, p)$, it must be the case that: $v^i(S_i) - p(S_i) \geq v^i(S_i^*) - p(S_i^*)$. Summing for all i and observing that $\sum_i p(S_i) = p([n]) = \sum_i p(S_i^*)$, we conclude that $\sum_i v^i(S_i) \geq \sum_i v^i(S_i^*)$.

The analogy with linear programming is completed by what is called the Second Welfare Theorem. It states that if $(p, S_1, \dots, S_m)$ is a Walrasian equilibrium and $S_1^*, \dots, S_m^*$ maximizes $\sum_i v^i(S_i^*)$, then $(p, S_1^*, \dots, S_m^*)$ is also a Walrasian equilibrium. The proof is also simple, observe that summing $v^i(S_i) - p(S_i) \geq v^i(S_i^*) - p(S_i^*)$ for all i we obtain $\sum_i v^i(S_i) \geq \sum_i v^i(S_i^*)$. But since this is an equality, we should have an equality for each agent i: $v^i(S_i) - p(S_i) = v^i(S_i^*) - p(S_i^*)$, hence $S_i^* \in D(v^i, p)$.

A natural question is for which economies there exist Walrasian equilibria. Kelso and Crawford define a very natural price adjustment procedure and define gross substitutes as the natural sufficient condition for such process to converge. The general idea behind this procedure goes back to Walras' *tatônnement procedure* (Walras, 2003), where tatônnement means *trial-and-error*. The idea is that we start with an arbitrary price vector and compute one set in the demand of each agent. Then, for each item that is demanded by more than one agent (over-demanded) we increase the price. For each item that is demanded by no agent (under-demanded), we decrease the price. We iterate this until no item is over-demanded or under-demanded.

Let's describe this procedure precisely. We will make some modifications to the idea above to make the procedure simpler to analyze. Instead of starting from an arbitrary price vector p, we will start with zero prices for all items and only allow prices to increase. Moreover, we will start with all the items allocated to the first player at zero price and we will take turns asking buyers to choose their favorite set of items given prices as follows: the current price p_j for items currently allocated to him and $p_j + \delta$ for items allocated to other players. Once he takes items from other players, the prices of such items increase by δ.

ALGORITHM 1: Walrasian tatônnement procedure.

Input: $\delta > 0$, $n, m \in \mathbb{Z}_+$ and v^i for $i \in [m]$
Set zero prices for all items: $p_j = 0, \forall j \in [n]$
Set initial allocation $S_1 = [n]$, $S_i = \emptyset, \forall i \in [m] \setminus \{1\}$
Implicitly define $p^i \in \mathbb{R}^n$ as a function of p s.t. $p^i_j = p_j$ if $j \in S_i$ and $p^i_j = p_j + \delta$ o.w.
while there exists i such that $S_i \notin D(v^i, p^i)$
  find a demanded set under the p^i price vector $X_i \in D(v^i, p^i)$
  update prices: for $j \in X_i \setminus S_i$, set $p_j = p_j + \delta$ (vectors p^i are implicitly updated)
  update allocations: $S_i = X_i$ and $S_j = S_j \setminus X_i$ for $j \neq i$

Notice that the procedure has to stop at some point, since prices cannot increase indefinitely. If the price of an item is higher then $\max_{i,S} v^i(S)$, for example, no agent will demand this item and the price will freeze. Let p be the final price and p^i be the price faces by each agent. It should be the case that $S_i \in D(v^i, p^i)$, which means that for all $T \subseteq [n]$, $v^i(S_i) - p^i(S_i) \geq v^i(T) - p^i(T)$. This can be re-written as: $v^i(S_i) - p(S_i) \geq v^i(T) - p(T) - \delta|T \setminus S_i|$.

In the limit as $\delta \to 0$, we recover a price vector and allocation such that $v^i(S_i) - p(S_i) \geq v^i(T) - p(T)$. To make the previous statement precise, let $(p^t, S_1^t, \dots, S_m^t)$ be the outcome of the Walrasian tatônnement procedure for $\delta_t = \frac{1}{t}$ for $t \in \mathbb{Z}_+$. Since there are finitely many allocations $(S_1^t, \dots, S_m^t)$, there is one allocation that happens infinitely often. Let $S_1, \dots, S_m$ be such allocation and let $t_1 < t_2 < \dots$ be the infinite subsequence corresponding to this allocation. Since p^t is bounded, passing to a subsequence if necessary, we can assume that $p^t \to p$. So taking $t \to \infty$ for this subsequence, we get $v^i(S_i) - p(S_i) \geq v^i(T) - p(T)$ for all i and $T \subseteq [n]$.

[2] The definition of Walrasian equilibrium can be changed to incorporate valuations not satisfying free-disposal. We do so by partitioning the items in $m+1$ disjoint sets $S_0, S_1, \dots, S_m$. The items in S_0 are not allocated and are required to be priced at zero at in equilibrium.