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On fair reinsurance premiums; Capital injections in a perturbed risk model

Zied Ben Salah [a], José Garrido [b,*]

[a] Department of Mathematics and Actuarial Sciences, American University in Cairo, P.O. Box 74, New Cairo 11835, Egypt
[b] Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd W, Montreal, Quebec H3G 1M8, Canada

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ABSTRACT

We consider a risk model where deficits after ruin are covered by a new type of reinsurance contract that provides capital injections. To allow the insurance company's survival after ruin, the reinsurer injects capital only at ruin times caused by jumps larger than a chosen retention level. Otherwise capital must be raised from the shareholders for small deficits. The problem here is to determine adequate reinsurance premiums. It seems fair to base the net reinsurance premium on the discounted expected value of any future capital injections. Inspired by the results of Huzak et al. (2004) and Ben Salah (2014) on successive ruin events, we show that an explicit formula for these reinsurance premiums exists in a setting where aggregate claims are modeled by a subordinator and a Brownian perturbation. Here ruin events are due either to Brownian oscillations or jumps and reinsurance capital injections only apply in the latter case. The results are illustrated explicitly for two specific risk models and in some numerical examples.

1. Introduction

Reinsurance contracts between a direct insurer and a reinsurer are used to transfer part of the risks assumed by the insurer. The problematic risks are those carrying either the possible occurrence of very large individual losses, the possible accumulation of many losses from non-independent risks, or those from other occurrences that could prevent insurers from fulfilling their solvency requirements. So traditionally reinsurance has been an integral part of insurance risk management strategies (see Centeno and Simões, 2009 for a survey of the different types of reinsurance and recent optimal reinsurance results). However, over time, global financial markets have developed additional or alternative risk transfer mechanisms, such as swaps, catastrophe bonds or other derivative products, that have helped insurers reduce their risk mitigation costs.

In this spirit of designing possibly cheaper risk transfer agreements we consider here a new type of reinsurance contract that would provide capital injections only in extreme, worse scenario cases. It differs from excess-of-loss (XL) agreements, or even catastrophe XL (Cat XL), in that it is neither a per-risk nor a per-event reinsurance contract, but rather one based on the insurer's financial position. Here ruin will serve as a simplifying proxy for the insurer's financial health. Reinsurance capital injections, after ruin, would allow the insurance company to continue to operate until the next ruin. Again to simplify the analysis we adopt an on-going concern basis and set an infinite horizon for the reinsurance treaty, which can allow repeated ruin events. The reinsurance agreement then calls for a capital injection after these successive ruin events, keeping the insurer afloat in perpetuity. We call this new type of agreement *reinsurance by capital injections* (RCI).

Here our jump–diffusion surplus process can generate two types of ruin events, hence different covers are assumed with distinct sources of capital. Surplus fluctuations due to jumps are assumed to represent larger claim costs from events unfavorable to the insurer; a ruin caused by such jumps will trigger a capital injection from the extreme-loss reinsurance contract, at ruin time, if the capital injection is larger than a certain threshold (retention limit). By contrast, Brownian oscillations represent comparatively smaller surplus fluctuations; so ruin caused by oscillations should be easier to cover with capital raised directly from the stockholders. Hence the reinsurer does not provide capital injections in cases when (1) ruin is from an oscillation, or (2) when it is from a jump producing a capital injection smaller than the threshold. As explained in the paper, even if stockholders may need cover these 2 types of ruin costs at first, they may ultimately get reimbursed by the reinsurer, at a subsequent ruin time due to a jump, if the latter is deep enough to meet the threshold.

Two recent developments in the literature make the analysis of the RCI contracts now possible, in the sense of getting tractable

* Corresponding author.
E-mail addresses: zied.bensalah@aucegypt.edu (Z. Ben Salah), jose.garrido@concordia.ca (J. Garrido).

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formulas for net premiums that would be fair to both parties for such agreements. The first one is the development of actuarial and financial models for capital injections (see for instance Einsenberg and Schmidli, 2011, or more recently Avram and Loke, 2018, and the references therein) and the other is the derivation of tractable formulas for the expected present value of future capital injections in a quite general class of risk models (see Huzak et al., 2004, and Ben Salah, 2014). The application presented here builds on this recent theory to develop fair lump sum net premiums for two types of RCI contracts, both over an infinite horizon. In practice our net premiums would have to be allocated to finite policy terms (e.g. a year) and loaded appropriately to define gross (market) premiums. In this first study we focus on the definition of the RCI contracts and the derivation of the premium formulas so that both, insurance and reinsurance companies, can compare the cost of RCI contracts to their alternative risk mitigation strategies/products. Future work would then need to address the issue of optimizing the insurance firm value by weighing these premiums in relation to other concurrent capital injections from shareholders.

To sum up, the paper is organized as follows: the general risk model used here is defined in Section 2. Then Section 3 covers the preliminary technical results needed to derive the expected present value of future capital injections. Section 4 gives the main result, with the derivation of fair premiums for reinsurance based on capital injections in the general risk model defined in Section 2. These are illustrated in detail for two classical risk processes in Section 5, which gives also numerical illustrations. The article concludes with some general remarks.

2. Risk model

We consider a general insurance surplus model that extends the standard Cramér–Lundberg theory to allow for jumps and diffusion type fluctuations. Here

$$R_t := x - Y_t\,, \qquad t \geqslant 0, \tag{2.1}$$

where $x \geqslant 0$ is the initial surplus and the risk process Y, a spectrally positive Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, is given by

$$Y_t := -c\,t + S_t + \sigma B_t\,, \qquad t \geqslant 0, \tag{2.2}$$

where $S = (S_t)_{t\geq 0}$ is a subordinator (i.e. a Lévy process of bounded variation and non-decreasing paths) without a drift ($S_0 = Y_0 = 0$) and B is a standard Brownian motion independent of S. Let ν be the Lévy measure of S; that is, ν is a σ-finite measure on $(0, \infty)$ satisfying $\int_{(0,\infty)}(1\wedge y)\nu(dy) < \infty$. In this case the Laplace exponent of S is defined by

$$\psi_S(s) = \int_{(0,\infty)} (e^{s\,y} - 1)\,\nu(dy),$$

where $\mathbb{E}[e^{sS_t}] = e^{t\,\psi_S(s)}$.

Note that the risk process in (2.1) is similar in spirit to the original perturbed surplus process introduced in Dufresne and Gerber (1991). The constant $x > 0$ represents the initial surplus, while the process Y models the cash outflow of the primary insurer and the subordinator S represents aggregate claims. That is why S needs to be an increasing process, with the jumps representing the claim amounts paid out. The Brownian motion B accounts for any small fluctuations affecting other components of the risk process dynamics, such as the claim arrivals, premium income or investment returns.

Here $c\,t$ represents aggregate premium inflow over the interval of time $[0, t]$. The premium rate c is assumed to satisfy the net profit condition, more precisely $\mathbb{E}[S_1] < c$, which means that

$$\int_{(0,\infty)} y\,\nu(dy) < c. \tag{2.3}$$

Condition (2.3) implies that the process Y has a negative drift, in order to avoid the possibility that R becomes negative almost surely. This condition is often expressed in terms of a safety loading applied to the net premium. For instance, note that we can recover the classical Cramér–Lundberg model if $\sigma = 0$ and $c := (1 + \theta)\mathbb{E}[S_1]$, for S a compound Poisson process modeling aggregate claims.

We do not use the concept of safety loading in this paper, in order to simplify the notation, but we stress the fact that this concept is implicitly considered within the drift of Y when we impose condition (2.3). The classical compound Poisson model is a special case of this framework where $\nu(dy) = \lambda\,K(dy)$, with λ being the Poisson arrival rate and K a diffuse claim distribution. We refer to Asmussen and Albrecher (2010) for an account on the classical risk model, and to Dufresne and Gerber (1991), Dufresne et al. (1991), Furrer and Schmidli (1994), Yang and Zhang (2001), Biffis and Morales (2010) and Ben Salah (2014) for the original and different generalizations or studies of the model in (2.2).

Now, one of the main objectives of this paper is to obtain an expression for the reinsurance premium for the risk model in (2.2). First we need to define quantities and notation associated with the ruin time, as well as the sequence of times of successive deficits due to a claim of the surplus process (2.2) after ruin. Let τ_x be the *ruin time* representing the first passage time of R_t below zero when $R_0 = x$, i.e.

$$\tau_x := \inf\{t > 0\,:\, Y_t > x\}, \tag{2.4}$$

where we set $\tau_x = +\infty$ if $R_t \geq 0$, for all $t \geq 0$. We define the first new record time of the running supremum

$$\tau := \inf\{t > 0\,:\, Y_t > \overline{Y}_{t-}\}, \tag{2.5}$$

and the sequence of times corresponding to new records of Y (that is $\overline{Y}_t := \sup\{Y_s\,:\, t \geq s\}$) due to a jump of S after the ruin time τ_x. More precisely, let

$$\tau^{(1)} := \tau_x, \tag{2.6}$$

and, assuming that $\{\tau^{(n)} < \infty\}$, then by induction on $n \geq 1$:

$$\tau^{(n+1)} := \inf\{t > \tau^{(n)}\,:\, Y_t > \overline{Y}_{t-}\}\,, \tag{2.7}$$

(note that by this definition $\tau^{(1)}$ differs from the consecutive new record times $(\tau^{(n)})_{n>1}$; the former includes ruin events caused by jumps and Brownian oscillations, while the latter include subsequent records only due to jumps).

Recall from Theorem 4.1 of Huzak et al. (2004) that the sequence $(\tau^{(n)})_{n>1}$ is discrete, and, in particular, neither time 0 nor any other time is an accumulation point of these $\tau^{(n)}$'s. More precisely, $\tau > 0$ a.s. and $\tau^{(n)} < \tau^{(n+1)}$ a.s. if $\{\tau^{(n)} < \infty\}$. As a consequence, we can order the sequence $(\tau^{(n)})_{n\geq 1}$ of times when a new supremum is reached by a jump of a subordinator as $0 < \tau^{(1)} < \tau^{(2)} < \cdots$ a.s.; see Fig. 1.

Finally, consider the random number

$$N := \max\{n : \tau^{(n)} < \infty\}, \tag{2.8}$$

which represents the number of new records reached by a claim of the surplus process in (2.2).

Before developing fair premiums for these new reinsurance by capital injections contracts that we define here, the next section first presents the theory available for the spectrally negative Lévy risk model defined in (2.2); see Doney and Kyprianou (2006), Kyprianou (2006) and Avram et al. (2007) for more details.