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Optimal risk allocation in reinsurance networks

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ABSTRACT

In this paper we consider reinsurance or risk sharing from a macroeconomic point of view. Our aim is to find socially optimal reinsurance treaties. In our setting we assume that there are n insurance companies, each bearing a certain risk, and one representative reinsurer. The optimization problem is to minimize the sum of all capital requirements of the insurers where we assume that all insurance companies use a form of Range-Value-at-Risk. We show that in case all insurers use Value-at-Risk and the reinsurer's premium principle satisfies monotonicity, then layer reinsurance treaties are socially optimal. For this result we do not need any dependence structure between the risks. In the general setting with Range-Value-at-Risk we obtain again the optimality of layer reinsurance treaties under further assumptions, in particular under the assumption that the individual risks are positively dependent through the stochastic ordering. Our results include the findings in Chi and Tan (2013) in the special case $n = 1$. At the end, we discuss the difference between socially optimal reinsurance treaties and individually optimal ones by looking at a number of special cases.

1. Introduction and motivation

Finding the optimal form of a reinsurance treaty or in more general terms optimizing risk sharing, is an old topic which regained a lot of attention in recent years. One of the first starting points has been Borch (1960) who proved that a stop-loss reinsurance treaty minimizes the variance of the retained loss of the insurer given the reinsurance premium is calculated with the expected value principle. A similar result has been derived in Arrow (1963) where the expected utility of terminal wealth of the insurer has been maximized. Since then a lot of generalizations of this problem have been considered. We refer the interested reader to the recent book Albrecher et al. (2017) which contains a comprehensive literature overview in chapter 8 and to Deelstra and Plantin (2014). We will here only mention a few recent articles which are relevant for our study. First, in Balbàs et al. (2009) a characterization of optimal reinsurance forms for a general class of risk measures has been given by exploiting duality theory in functional analysis. A stop-loss treaty turned out to be optimal when the premium principle is an expected value principle. Further, Chi and Tan (2013) considered the optimization problem with Value-at-Risk and Expected Shortfall and a general premium principle for the reinsurer. They obtained the optimality of a layer-reinsurance.

While most publications consider the problem only from the perspective of the individual insurer, we investigate the situation from an economic point of view. More precisely, we want to know what kind of risk sharing between insurers and reinsurer is optimal for the entire economy and in which situations it is identical to the individually optimal decisions of the insurers. This question also makes it necessary to address the task of modelling the problem for a random vector representing the individual risks taken by the insurers. There exists of course a rich literature on risk sharing problems where random vectors are involved. The most popular problem is the so-called *inf-convolution* problem which is given by

$$\min \sum_{i=1}^{n} \rho_i(X_i) \quad \text{s.t. } X_1 + \cdots + X_n = X$$

where ρ_i are suitable risk measures and $X_1, \ldots, X_n, X$ are random variables representing risks which are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It has been shown in Filipović and Svindland (2008) that for law- and cash-invariant convex risk measures a solution always exists and is given by a comonotone structure. This result has been refined by Embrechts et al. (2018) where it has been shown that if the risk measures are given by Range-Value-at-Risk, there is an explicit construction for the optimal solution. In Kiesel and Rüschendorf (2013) this problem has been interpreted in a setting with several insurers with general convex risk measures and premium principles. There, optimal reinsurance contracts have been characterized by means of subdifferential formulas in Banach spaces. For more results on the inf-convolution problem we refer the reader to Rüschendorf (2013).

Problems where special kinds of risk sharing between two entities are considered can be found e.g. in Asimit et al. (2013).

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There, the insurance group allocates the total risk between two entities which are subject to different regulatory capital requirements, using appropriate risk transfer agreements. The optimal risk sharing rule is derived explicitly for special risk measures like Value-at-Risk and Expected Shortfall. In Cai et al. (2016) the authors develop optimal reinsurance contracts that minimize the convex combination of the Value-at-Risk of the insurer's loss and the reinsurer's loss under some constraints. Some explicit, though rather complicated optimal reinsurance treaties are obtained there. Next, Chi and Meng (2014) investigate the optimal form of reinsurance from the perspective of an insurer when he decides to cede parts of the loss to two reinsurers, where both reinsurers calculate the premium according to different premium principles. The problem is solved under the criterion of minimizing Value-at-Risk or Expected Shortfall. An optimal reinsurance treaty is to cede two adjacent layers of the risk. Another multivariate problem is considered in Zhu et al. (2014) where optimal reinsurance strategies for an insurer with multiple lines of business are investigated under the criterion of minimizing the total capital requirement calculated based on the multivariate lower-orthant Value-at-Risk. The optimal strategy for the insurer there is to buy a two-layer reinsurance treaty for each line of business. Note that the dependence structure for the individual risks was not important for the results cited so far. A worst case scenario w.r.t. the dependence structure has been considered in Cheung et al. (2014) where the problem of optimal reinsurance treaties for multivariate risks with general law-invariant convex risk measures has been studied. It turned out that stop-loss reinsurance treaties minimize a general law-invariant convex risk measure of the total retained risk. In Cai and Wei (2012) it has been assumed that an insurer has n lines of business which can be reinsured subject to a given premium and the aim is to minimize an expected convex function of the total retained risk. In order to derive results in this setting the authors needed a concept for positive dependence between risks which has been the concept of 'positively dependent through the stochastic ordering'.

Papers with a more economic point of view on optimal reinsurance are among others d'Ursel and Lauwers (1985) where a Stackelberg equilibrium for n reinsurers under special assumptions is considered and Powers and Shubik (2001) where a game-theoretic analysis of optimal insurance networks has been conducted.

The aim of this paper now is to consider reinsurance or risk sharing from a macroeconomic point of view. Whereas the individual goal of an insurance company is to reduce risk exposure and own capital requirements by reinsurance, the social goal of reinsurance is to spread risk around the globe to avoid local overexposures. This construction also increases the amount of risk which can be insured. In our setting we assume that there are n insurance companies, each bearing a certain risk, and one representative reinsurer. In contrast to the inf-convolution problem the situation is no longer symmetric. The optimization problem then is to minimize the sum of all capital requirements of the insurers. We assume that all insurance companies use Range-Value-at-Risk as a risk measure with possibly different parameters. Range-Value-at-Risk comprises Value-at-Risk and Expected Shortfall and is thus a natural choice with practical relevance, see e.g. Cont et al. (2010). We show that in case all insurers use Value-at-Risk and the reinsurer's premium principle satisfies monotonicity, then layer reinsurance treaties are socially optimal. For this result we do not need any dependence structure between the risks. In the general setting with Range-Value-at-Risk we obtain again the optimality of layer reinsurance treaties under the assumption that the reinsurer's premium principle is consistent with the increasing convex order (which most premium principles are) and under the assumption that the individual risks are positively dependent through the stochastic ordering (PDS). Our results include the findings in Chi and Tan (2013) in the special case $n = 1$. Finally, we also discuss the difference between socially optimal reinsurance treaties and individually optimal ones. Fortunately, they coincide in many cases but there also may be some differences.

Our paper is organized as follows: In the Section 2 we summarize some definitions and facts from risk measures, stochastic orders and dependence concepts. In particular, we prove that PDS random vectors carry the increasing convex order of the margins over to the sum of the components. In Section 3 we introduce and discuss our optimization problem. The solution of the problem is then presented in Section 4 where also some special cases are discussed. In Section 5 we investigate the difference between socially optimal reinsurance treaties and individually optimal ones by looking at a number of special cases.

2. Risk measures, stochastic orders and dependence structures

We will consider non-negative random variables $X : \Omega \to \mathbb{R}_+$ defined on a non-atomic probability space $(\Omega, \mathcal{A}, \mathbb{P})$. They represent future insurance claims, i.e. $X(\omega) \geq 0$ is the discounted net loss of an insurance company at the end of a fixed period. We denote the (cumulative) distribution function by $F_X(x) := \mathbb{P}(X \leq x)$, the survival function by $S_X(x) := 1 - F_X(x)$ and the generalized inverse by $F_X^{-1}(\alpha) := \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$ where $x \in \mathbb{R}$ and $\alpha \in [0, 1]$. With

$$L^1 := \{X : \Omega \to \mathbb{R}_+ : X \text{ is a random variable with } \mathbb{E}[X] < \infty\}$$

we denote the space of all such non-negative, integrable random variables. We now recall some notions of risk measures. In general, a risk measure is a mapping $\rho : L^1 \to \bar{\mathbb{R}}$. Essentially, the notion of a premium principle $\pi : L^1 \to \bar{\mathbb{R}}$ is mathematically equivalent but applications are different. While the former determines the necessary solvency capital to bear a risk, the latter gives the price of (re)insuring it. The properties of risk measures discussed in the sequel apply to premium principles analogously. Of particular importance are the following risk measures.

Definition 2.1. For $\alpha, \beta \in [0, 1]$ and $X \in L^1$ with distribution function F_X we define

(a) the *Value-at-Risk* of X at level α as $VaR_\alpha(X) := F_X^{-1}(1-\alpha)$.
(b) the *Expected Shortfall* of X at level $\beta > 0$ as $ES_\beta(X) := \frac{1}{\beta}\int_0^\beta VaR_s(X)ds$.
(c) the *Range-Value-at-Risk* of X at level α, β if $\alpha + \beta \leq 1$ as

$$RVaR_{\alpha,\beta}(X) := \begin{cases} \dfrac{1}{\beta}\displaystyle\int_\alpha^{\alpha+\beta} VaR_s(X)ds, & \beta > 0 \\ VaR_\alpha(X), & \beta = 0. \end{cases}$$

Obviously, Range-Value-at-Risk comprises both Value-at-Risk and Expected Shortfall.
A risk measure ρ should have some nice properties like for example

(i) *law-invariance:* $\rho(X)$ depends only on the distribution F_X.
(ii) *monotonicity:* If $X \leq Y$ then $\rho(X) \leq \rho(Y)$.
(iii) *translation invariance:* For $m \in \mathbb{R}$ it holds $\rho(X + m) = \rho(X) + m$.
(iv) *positive homogeneity:* For $\alpha \geq 0$ it holds that $\rho(\alpha X) = \alpha\rho(X)$.
(v) *subadditivity:* $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
(vi) *convexity:* For $\alpha \in [0, 1]$ it holds that $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$.

Though Value-at-Risk is in general not subadditive it has a lot of nice properties like law-invariance, monotonicity, translation invariance and positive homogeneity. These properties are also shared by Range-Value-at-Risk. The following facts, which can be directly derived from the definition, will be important for us: