Insurance: Mathematics and Economics 82 (2018) 141–151

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Upper bounds for strictly concave distortion risk measures on moment spaces

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ARTICLE INFO

Article history:
Received January 2018
Received in revised form July 2018
Accepted 10 July 2018

JEL classification:
C60

Keywords:
Value-at-Risk (vaR)
Coherent risk measure
Model uncertainty
Cantelli bound
Distortion function

ABSTRACT

The study of worst-case scenarios for risk measures (e.g., Value-at-Risk) when the underlying risk (or portfolio of risks) is not completely specified is a central topic in the literature on robust risk measurement. In this paper, we tackle the open problem of deriving upper bounds for strictly concave distortion risk measures on moment spaces. Building on early results of Rustagi (1957, 1976), we show that in general this problem can be reduced to a parametric optimization problem. We completely specify the sharp upper bound (and corresponding maximizing distribution function) when the first moment and any other higher moment are fixed. Specifically, in the case of a fixed mean and variance, we generalize the Cantelli bound for (Tail) Value-at-Risk in that we express the sharp upper bound for a strictly concave distorted expectation as a weighted sum of the mean and standard deviation.

0. Introduction

In this paper, we study upper bounds on the risk of a portfolio with respect to a (strictly) concave distortion risk measure when the underlying risk is not fully specified in that only some information on its moments is available. This problem is relevant for several reasons. *First*, a concave distortion risk measure is coherent (Artzner et al., 1999) and thus has all properties that "good" risk measures are typically expected to have. Moreover, if in addition to the coherency of a law-invariant[1] risk measure, one also requires comonotone additivity, then concave distortion risk measures are the only ones that are admissible (Kusuoka, 2001). *Second*, measuring the risk of portfolios is at the center of insurance activities. When the marginal distribution functions of the portfolio components and their dependence structure are both known, the risk of the portfolio can be assessed numerically, for instance by using Monte-Carlo simulation. In most cases, however, it cannot be expected that full information on the dependence structure is available, and various stakeholders such as investors and regulators could be interested in the worst-case scenario for the portfolio (i.e., when the risk measure attains its highest value). In this regard, we note that there is a rich literature on finding bounds for the Value-at-Risk (VaR) of a portfolio under the assumption that all marginal distribution functions are known, but the dependence is either unknown or only partially known.[2] In this paper, however, we do not fix the marginal distribution functions, but derive bounds under the sole knowledge of some moments of the portfolio loss (e.g. based on portfolio statistics) without specification of the marginal distribution functions. Moreover, we consider the class of strictly concave distortion risk measures, and the VaR does not belong to this class.

The best-known concave distortion risk measure is the Tail Value-at-Risk (TVaR), also called Expected Shortfall in the literature. This measure quantifies the expected value of the risk given that it is greater than its Value-at-Risk (measured at the same probability level). In fact, TVaR is the smallest coherent risk measure that is greater than the Value-at-Risk (VaR), which is the most frequently used risk measure in risk management and supervision practice, but which fails to be subadditive and thus lacks coherency. Effectively, the VaR is a particular quantile of the

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1 There is an increasing interest in the study of risk measures that account for background risk; see e.g. Mainik and Schaanning (2014) and Sordo et al. (2018).

2 Papers relevant here include Rüschendorf (1982), Denuit et al. (1999a), Kaas et al. (2009), Wang and Wang (2011), Wang et al. (2013), Embrechts et al. (2013, 2014, 2015a), Hofert et al. (2017), Puccetti et al. (2016, 2017), Bernard et al. (2015, 2017) and Rüschendorf and Witting (2017).

https://doi.org/10.1016/j.insmatheco.2018.07.002
0167-6687/

distribution, whereas TVaR[3] is more focused on the right tail of the distribution in that it measures the expected loss, conditionally on the loss being greater than VaR. Moment bounds for VaR (which are intimately connected to distributional bounds) and TVaR have already been studied in the insurance literature by authors such as Kaas and Goovaerts (1986), Denuit et al. (1999b), De Schepper and Heijnen (2010), Hürlimann (2002, 2008), Goovaerts et al. (2011), Bernard et al. (2015, 2016) and Tian (2008). Specifically, Hürlimann (2002) finds analytical bounds for VaR and TVaR under knowledge of the mean, variance, skewness and kurtosis. An elementary derivation of bounds on VaR can be found in Bernard et al. (2016). In this regard, we point out that one cannot expect that there exists a risk measure (i.e., a single number) that captures all characteristics of risk and provides a complete picture of the risky portfolio (i.e., a random variable). For example, Hürlimann (2002) studies TVaR for various two-parameter distribution functions with fixed mean and variance by varying the loss probability and argues that TVaR does not always properly reflect the increase in (tail) risk from one distribution to another. Moreover, risk measures appear in various contexts, such as risk management (McNeil et al., 2015), pricing (Wirch and Hardy, 1999), capital allocation (Dhaene et al., 2012) and supervision (Danielsson et al., 2001), and a risk measure that is suitable for one purpose might be not appropriate in another context; see also Dhaene et al. (2008) for a warning against the blind use of coherent risk measures as well as Belles-Sampera et al. (2014a, b), Frittelli et al. (2014), Bellini et al. (2014), Kou and Peng (2016) and Cai et al. (2017) for recent proposals of risk measures.

In this paper, we study bounds for any strictly concave distortion risk measure when k, not necessarily consecutive, moments of the underlying risk are known. This problem can be cast as an extended version of an optimization problem considered in Rustagi (1957, 1976). This author considers the optimization of a certain integral when the first moment and second moment are known and provides some necessary conditions its solution must satisfy. *Our first contribution* is to show that optimization of concave distortion risk measures is compatible with this integral formulation and to provide, for an arbitrary sequence of moments, the necessary conditions a solution has to satisfy. In this regard, Rustagi (1957, 1976) claims that in certain cases the necessary conditions he derives lead to complete specification of the solution, but a proof is missing and appears to be non-trivial; we provide a proof in Section 2.2. *Our second contribution* is to completely specify the maximizing distribution function (worst-case scenario) when the mean and any other higher moment are known and to provide an algorithm to obtain it. As a *third contribution*, we derive, in the specific case of fixed mean and standard deviation, a Cantelli like formula[4] in that we are able to express the worst-case distorted expectation as a weighted sum of the mean and standard deviation. Such a formula is of potential interest for robust portfolio optimization. Specifically, Ghaoui et al. (2003) deal with the problem of finding a portfolio that optimizes the worst-case VaR when the distribution of returns is only partially known (only mean and covariance matrix are available). In their analysis, the Cantelli bound is crucial in that it essentially allows reformulation of their optimization problem as a mean–variance optimization problem à la Markowitz. Using the generalized version of the Cantelli bound, this approach to robust portfolio selection can be extended to any concave distorted expectation. Finally, our *fourth contribution* is to obtain useful (although non-sharp) bounds for the general case.

[3] There are various proofs that demonstrate subadditivity of TVaR; see Embrechts et al. (2015b) and references herein.

[4] The Cantelli bound expresses worst-case VaR as a weighted sum of (known) mean and standard deviation.

1. Problem formulation

In this paper, we mainly consider distribution functions with a bounded domain. Hence, after rescaling, we consider them on the unit interval $[0, 1]$. The restriction to bounded domains may appear as a limitation of the setting but is actually often appropriate, in particular in a risk management context within financial institutions. In this context, one is typically concerned only with losses, not by gains; i.e., a lower bound of zero applies. The maximum loss a bank can suffer on its mortgage portfolio is bounded by the total amount that is lent. The upper limit to the financial loss for which the insurance company underwrites is generally fixed by the contract or determined through reinsurance techniques; moreover, insurers have limited liability (up to their capital) to meet claims.

Denote by $\mathcal{F}$ a set of distribution functions on $[0, 1]$ for which $k \in \mathbb{N}_0$ moments are given,

$$\mathcal{F} = \left\{ F \text{ is a cdf on } [0,1] \,\middle|\, \int_0^1 x^i dF(x) = c_i,\, i \in \mathcal{I} \right\} \\ := \mathcal{F}((c_i)_{i\in\mathcal{I}}), \tag{1}$$

where $\mathcal{I} \subset \mathbb{N}_0$ and $card(\mathcal{I}) = k$. Note that in general $\mathcal{F}$ may correspond to a set of distribution functions with any k moments fixed, not necessarily the first k ones, and not necessarily starting with the mean. In the remainder of the paper, we assume that $\mathcal{F}$ contains at least two different elements (and hence infinitely many, since $\mathcal{F}$ is convex).

A distortion risk measure of a random variable X having cumulative distribution F_X is defined as

$$H_g(X) = \int_0^{\infty} g(1 - F_X(x))dx - \int_{-\infty}^{0} [1 - g(1 - F_X(x))]\, dx, \tag{2}$$

where g is a distortion function, i.e., an increasing function on $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$. Note that $H_g(X)$ depends solely on the distribution function F_X (law-invariance), and in what follows we also write $H_g(F_X)$ instead of $H_g(X)$. Furthermore, we assume that g is strictly concave and twice differentiable, implying that H_g is a coherent risk measure; see e.g., Dhaene et al. (2006). The importance of distortion risk measures with concave distortion function (henceforth called concave distortion risk measures) is highlighted by the fact that this class coincides with the class of coherent risk measures that are law-invariant and comonotone additive (Kusuoka, 2001). Examples of concave distortion risk measures are the power distortion risk measure ($g(x) = x^{\alpha}$, $\alpha \in (0, 1)$), the dual-power distortion risk measure ($g(x) = 1-(1-x)^{\beta}$, $\beta \in (1, \infty)$) and the Wang distortion risk measure ($g(x) = \Phi(\Phi^{-1}(x)+\Phi^{-1}(p))$, $p \in (0.5, 1)$).

In this paper, we focus on the problem of determining the distribution function in $\mathcal{F}$ that yields maximum (concave) distorted expectation, i.e., we consider the optimization problem

$$\sup_{F\in\mathcal{F}} H_g(F). \tag{3}$$

When only one moment is specified, say the ith one with value c_i, it is easy to show that the solution is obtained by a discrete distribution function F that is concentrated on 0 and 1 and has ith moment equal to c_i. To see this, observe that F dominates all other admissible distribution functions in the sense of stop-loss order (since F crosses all other distribution functions exactly once from above and has the biggest possible mean, namely c_i) and it is well-known that concave distortion risk measures are consistent with stop-loss order (see e.g., Dhaene et al. (2006)). Hence, the case $k = 1$ is not interesting and, moreover, since little distributional information is used in the optimization, this case is not very useful in practice in that it leads to wide bounds. Therefore, in the remainder of the paper we consider only the case in which $k \geqslant 2$.