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Banach Contraction Principle and ruin probabilities in regime-switching models

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ABSTRACT

We apply Banach Contraction Principle to approximate a vector $\boldsymbol{\Psi}$ of ruin probabilities in regime-switching models. A Markov chain is interpreted as a 'switch' that changes the amount and/or wait time distributions of claims. The insurer has a possibility to adapt the premium rates in response. An associated risk operator $\mathbf{L}$ is proven to be a contraction on a properly chosen complete metric space while $\boldsymbol{\Psi}$ is shown to be the unique fixed point of $\mathbf{L}$ within this space. Thus, by iterating $\mathbf{L}$ on any of its points, we can simultaneously approximate $\boldsymbol{\Psi}$ and control the error of approximation. Numerical examples confirm high accuracy of the resulting procedure.

1. Introduction

Several aspects of regime-switching risk processes have been studied recently (see e.g. Wang et al. (2016), Landriault et al. (2015), Chen et al. (2014), Guillou et al. (2013)). Although numerous monographs and papers deal with embedding the Cramér–Lundberg model into Markovian environment, see e.g. Asmussen and Albrecher (2010), the actuarial literature related to a *regime-switching Sparre Andersen model* is rather scarce. Since the topic is of great interest in applications, we will focus on it in this paper. Our aim is threefold:

1. We will provide a universal method of approximating the ultimate ruin probability $\boldsymbol{\Psi}(u)$ which is a vector of functions of the initial surplus u in this case.
2. The approximation error will be measured globally, with respect to all $u \geqslant 0$ simultaneously.
3. To achieve the above aims, we will provide a new methodology based on Banach Contraction Principle.

Banach Contraction Principle is a powerful tool to solve integral and differential equations, giving a constructive method to approximate their solutions with a controlled precision (see e.g. Kilbas et al. (2006)). However, its applications to evaluate ruin probabilities do not seem to be immediate. A simple cause is that an associated risk operator $\mathbf{L}$, given by (5), has usually infinitely many fixed points (see Remark 3.1 for details). In this paper, we will show how to make it a contraction on a properly defined complete metric space $\langle \mathcal{R}^s, d_r \rangle$ given by (11). As a result, a vector of ultimate ruin probabilities is shown to be the unique fixed point of $\mathbf{L}$ in $\langle \mathcal{R}^s, d_r \rangle$. What is more, we can approximate the ultimate ruin probabilities with controlled precision by iterating $\mathbf{L}$ on every starting point from $\langle \mathcal{R}^s, d_r \rangle$.

To be more precise, let all stochastic objects considered in the paper be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us denote by $\mathbb{N}$ the set of all positive integers. Let $\mathbb{R}$ denote the real line. Set $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_+^0 = [0, \infty)$ and $\overline{\mathbb{R}}_+ = (0, \infty]$. Let a random variable X_k denote the amount of the kth claim, T_1 – the moment when the first claim appears and T_k – the time between the $(k-1)$th claim and the kth one. We will denote by A_n the moment when the nth claim appears. With this notation, $A_n = T_1 + \cdots + T_n$ under the convention that $A_0 = 0$. Let a random variable C_k denote the insurance premium rate during the time interval $[A_{k-1}, A_k)$. Let $\{I_k\}_{k \in \mathbb{N}^0}$ be a homogeneous Markov chain with a finite state space $S = \{1, 2, \ldots, s\}$ such that the probabilities $p_i = \mathbb{P}(I_0 = i)$ are positive for every $i \in S$. A transition matrix $P = (p_{ij})_{i,j \in S}$ is such that the probabilities $p_{ij} = \mathbb{P}(I_{k+1} = j | I_k = i)$ are non-negative for all $i, j \in S$. The jump from I_{k-1} to I_k can change the distribution of T_k and/or X_k at the moment A_k only, so we will interpret $\{I_k\}_{k \in \mathbb{N}^0}$ as 'switches'. We assume that the insurance premium rate $C_k = c(I_{k-1})$, where c is a known positive function defined on S. The conditional distribution of X_1 (respectively T_1),

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given the initial state i and the state j at the moment A_1, will be denoted by F^{ij} (respectively G^{ij}), see Section 2 for details.

Let a non-negative real u denote the insurer's surplus at 0 and $U_n = U(n,u)$ — at the moment A_n, respectively. The *surplus process* (*risk process*) $\{U_n\}_{n\in\mathbb{N}}$ is defined by

$$U_n = u - \sum_{k=1}^{n}(X_k - c(I_{k-1})T_k)\,. \tag{1}$$

The *time of ruin*

$$\tau = \tau(u) = \inf\{n \in \mathbb{N} : U(n,u) < 0\} \tag{2}$$

is the first time when the insurer's surplus falls below zero (here $\inf\emptyset$ means ∞). The conditional probability that $\tau(u)$ is not greater than n, given the initial state i, considered as a function of the initial surplus u, is called the probability of ruin at or before the nth claim. Let us denote it by $\Psi_n^i(u)$ and write

$$\boldsymbol{\Psi}_n(u) = (\Psi_n^1(u), \dots, \Psi_n^s(u))\,. \tag{3}$$

The conditional probability that $\tau(u)$ is finite, given the initial state i, is called the *infinite horizon ruin probability* (or *ultimate ruin probability*). Let us denote it by $\Psi^i(u)$ and write

$$\boldsymbol{\Psi}(u) = (\Psi^1(u), \dots, \Psi^s(u))\,. \tag{4}$$

Let $\mathcal{R}$ denote the set of all measurable functions defined on non-negative reals and taking values in $[0,1]$ almost everywhere. We will use the symbol $\mathcal{R}^s$ to denote the set $\{(\rho_1,\dots,\rho_s) : \rho_i \in \mathcal{R}$ for every $i \in S\}$. The elements of $\mathcal{R}^s$ will be written in bold.

We call $\mathbf{L} : \mathcal{R}^s \to \mathcal{R}^s$ the *risk operator* if

$$\mathbf{L}\boldsymbol{\rho}(u) = (\mathrm{L}_1\boldsymbol{\rho}(u), \dots, \mathrm{L}_s\boldsymbol{\rho}(u)), \quad u \geqslant 0, \tag{5}$$

where, under the convention that $\int_a^\infty$ means $\int_{(a,\infty)}$,

$$\mathrm{L}_i\boldsymbol{\rho}(u) = \sum_{j=1}^{s} p_{ij}\int_0^\infty \int_{(0,\,u+c(i)t]} \rho_j(u + c(i)t - x)dF^{ij}(x)dG^{ij}(t) + \sum_{j=1}^{s} p_{ij}\int_0^\infty \int_{u+c(i)t}^\infty dF^{ij}(x)dG^{ij}(t), \quad i \in S\,. \tag{6}$$

An important relationship between $\boldsymbol{\Psi}_{n+1}$, $\boldsymbol{\Psi}_1$ and the risk operator $\mathbf{L}$ is given by the following equality:

$$\boldsymbol{\Psi}_{n+1}(u) = \mathbf{L}\boldsymbol{\Psi}_n(u) = \mathbf{L}^n\boldsymbol{\Psi}_1(u), \quad u \geqslant 0 \tag{7}$$

(see Theorem 3.1 for details). Let us denote

$$M^i(r) = \sum_{j=1}^{s} p_{ij}\int_0^\infty \int_0^\infty e^{-r(c(i)t-x)}dF^{ij}(x)dG^{ij}(t), \quad i \in S,\ r \in \mathbb{R}\,. \tag{8}$$

Positive constants $r_0^1, \dots, r_0^s$ will be called *adjustment coefficients* if

$$M^i(r_0^i) = 1, \quad i \in S\,. \tag{9}$$

A sufficient condition for the existence of the *adjustment vector* $(r_0^1, \dots, r_0^s)$ is given in Theorem A.1. For a fixed $r \in (0, \min_{i\in S}\{r_0^i\})$, let us define the following norm:

$$\|\boldsymbol{\rho}\|_r = \max_{i\in S}\{\int_0^\infty |\rho_i(u)|e^{ru}du\}, \quad \boldsymbol{\rho} \in \mathcal{R}^s\,. \tag{10}$$

As usual, the corresponding metric is defined by $d_r(\boldsymbol{\rho}^1, \boldsymbol{\rho}^2) = \|\boldsymbol{\rho}^1 - \boldsymbol{\rho}^2\|_r$.

Lemma 2.2 shows that the following subset of $\mathcal{R}^s$:

$$\langle\mathcal{R}^s, d_r\rangle = \{\boldsymbol{\rho} \in \mathcal{R}^s : \|\boldsymbol{\rho}\|_r < \infty\} \tag{11}$$

is a complete metric space. By Theorem 2.1, $\mathbf{L}$ is a contraction on $\langle\mathcal{R}^s, d_r\rangle$. Consequently,

$$\boldsymbol{\Psi} = \mathbf{L}\boldsymbol{\Psi}\,,$$

by Banach Contraction Principle. Additionally, consecutive iterations of $\mathbf{L}$ on any $\boldsymbol{\rho} \in \langle\mathcal{R}^s, d_r\rangle$ converge, in the sense of $\|\cdot\|_r$, to $\boldsymbol{\Psi}$ and

$$\|\mathbf{L}^n\boldsymbol{\rho} - \boldsymbol{\Psi}\|_r \leqslant \frac{[M^*(r)]^n}{1 - M^*(r)}\|\mathbf{L}\boldsymbol{\rho} - \boldsymbol{\rho}\|_r\,, \tag{12}$$

where $M^*(r) = \max_{i\in S}\{M^i(r)\}$ (see Theorem 3.2 for details). Moreover, (12) implies that

$$\max_{i\in S}\{\int_0^\infty |\mathrm{L}_i^n\boldsymbol{\rho}(u) - \Psi^i(u)|du\} \leqslant \inf_{r\in(0,\,r_0^*)}\left\{\frac{[M^*(r)]^n}{1 - M^*(r)}\|\mathbf{L}\boldsymbol{\rho} - \boldsymbol{\rho}\|_r\right\},$$

where $r_0^* = \min\{r_0^i : i \in S\}$ and $\mathrm{L}_i^n\boldsymbol{\rho}$ is the ith coordinate of $\mathbf{L}^n\boldsymbol{\rho}$ (see Corollary 3.2 for details). Numerical Examples 3.2–3.3 show that our methodology can lead to precise approximations of $\boldsymbol{\Psi}$.

The one-dimensional non-switching Sparre-Andersen risk model enables to treat by the same theory (see e.g. Thorin (1975, p. 88)) absolutely continuous as well as discrete distributions of the inter-arrival time. This idea applied to a regime-switching framework leads to a fairly general risk model in which one can approximate ruin probabilities $\boldsymbol{\Psi}$ no matter what time – continuous or discrete – is considered. Under specified assumptions (see discussion in Gajek and Rudź (2017)), the above regime-switching Sparre-Andersen model generalizes several continuous- and discrete-time risk models. Let us recall just a few of them: a regime-switching model with exponentially distributed inter-arrival times (cf. Example 3.1), a discrete time regime-switching model (see Example 3.2), a non-switching discrete time risk model (see e.g. Example 3.3), the Sparre Andersen model (see e.g. Sparre Andersen (1957)) and the classical non-switching Cramér–Lundberg model (see e.g. the below mentioned papers by Politis (2006) or Gordienko and Vázquez-Ortega (2016)).

For a survey of Markov switching models, we refer the reader to Frühwirth-Schnatter (2006). Markov additive processes are studied in Asmussen (2003) or Feng and Shimizu (2014) among others. Asmussen's (2003, p. 310) monography contains also a convenient simulation scheme for these models. The Markov-modulated Poisson process can be found in numerous articles and monographs, for instance, in Reinhard (1984), Asmussen (1989) or Asmussen and Albrecher (2010). Estimation of the parameters of a Markov-modulated insurance loss process is discussed in Guillou et al. (2013). Enikeeva et al. (2001) propose a method of continuity analysis of ruin probabilities with respect to variation of parameters governing risk processes and illustrate it by Markov modulated risk models as well as by the Sparre Andersen model. A risk model with a Markovian arrival process is investigated in Li et al. (2015). Xu et al. (2017) investigate optimal investment and reinsurance policies for an insurer under a Markov-modulated financial market. Wang et al. (2016) consider a multi-dimensional regime-switching risk model in which the surplus process for each class of insurance business is assumed to follow a compound Cox risk process. Landriault et al. (2015) propose a drawdown-based regime-switching Lévy insurance model in which the underlying drawdown process is used e.g. to model an insurer's level of financial distress over time. Chen et al. (2014) investigate a second order integro-differential system of equations for the expected discounted dividend payments in a Markov-modulated jump–diffusion risk model with randomized observation periods and threshold dividend strategy.

Taylor (1976) was perhaps the first researcher who used the operator approach to evaluate ruin probabilities. Since then, this approach has been further investigated and developed by Gajek (2005) and Gajek and Rudź (2013, 2017). In the classical