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Pricing insurance drawdown-type contracts with underlying Lévy assets

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ABSTRACT

In this paper we consider some insurance policies related to drawdown and drawup events of log-returns for an underlying asset modeled by a spectrally negative geometric Lévy process. We consider four contracts, three of which were introduced in Zhang (2013) for a geometric Brownian motion. The first one is an insurance contract where the protection buyer pays a constant premium until the drawdown of fixed size of log-returns occurs. In return he/she receives a certain insured amount at the drawdown epoch. The next insurance contract provides protection from any specified drawdown with a drawup contingency. This contract expires early if a certain fixed drawup event occurs prior to the fixed drawdown. The last two contracts are extensions of the previous ones by an additional cancellation feature which allows the investor to terminate the contract earlier. We focus on two problems: calculating the fair premium p for the basic contracts and identifying the optimal stopping rule for the policies with the cancellation feature. To do this we solve some two-sided exit problems related to drawdown and drawup of spectrally negative Lévy processes, which is of independent mathematical interest. We also heavily rely on the theory of optimal stopping.

1. Introduction

The drawdown of a given process is the distance of the current value away from the maximum value it has attained to date. Similarly, the drawup is defined as the current rise in the value over the running minimum. Both have been customarily used as dynamic risk measures. In fact, the drawdown process not only provides dynamic measure of risk, but it can also be viewed as giving measure of relative regret. Similarly the drawup process can be viewed as providing measure of relative satisfaction. Thus, a drawdown or a drawup may signal the time when the investor may choose to change his/her investment position, which depends on his/her perception of future moves of the market and his/her risk aversion.

The interest in the drawdown process has been strongly raised by the recent financial crisis. A large market drawdown may bring portfolio losses, liquidity shocks and even future recessions. Therefore risk management of drawdown has become so important among practitioners; see e.g. Grossman and Zhou (1993) for portfolio optimization under constraints on the drawdown process, Carr et al. (2011) and Magdon-Ismail and Atiya (2004) for the distribution of the maximum drawdown of drifted Brownian motion and the time-adjusted measure of performance known as the Calmar ratio, and (Pospisil and Vecer, 2010; Vecer, 2006, 2007) for the drawdown process as a dynamic measure of risk. For an overview of the existing techniques for analysis of market crashes as well as a collection of empirical studies of the drawdown process and the maximum drawdown process, see Sornette (2003).

It is thus natural that fund managers have a strong incentive to seek insurance against drawdown. In fact, as papers Carr et al. (2011), Vecer (2006) and Vecer (2007) argue, some market-traded contracts, such as vanilla and look-back puts, have only limited ability to insure against market drawdown. Therefore the drawdown protection can be useful also for individual investors.

In this paper we follow Zhang et al. (2013) in pricing some insurance contracts against drawdown (and drawup) of log-returns of stock price modeled by an exponential Lévy process and identifying the optimal stopping rules. We also identify for these contracts the so-called fair premium rates for which the contract prices equal zero.

In its simplest form, the first drawdown insurance contract involves a continuous premium payment by the investor (protection buyer) to insure against a drawdown of log-returns of the underlying asset over a pre-specified level. A possible buyer of this contract might think that a large drawdown is unlikely and he/she might want to stop paying the premium. Therefore we expand the simplest contract by adding a cancellation feature. In this case, the

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investor receives the right to terminate the contract earlier and in that case he/she pays a penalty for doing so. We show that the investor's optimal cancellation time is based on the first passage time of the drawdown of the log-return process.

Moreover, we also consider a related contract that protects the investor from a drawdown of log-return of the asset price preceding a drawup related to it. In other words, the contract expires early if a drawup occurs prior to a drawdown. From the investor's perspective, when a drawup occurs, there is little need to insure against a drawdown. Therefore, this drawup contingency automatically stops the premium payment, and it is an attractive feature that could potentially reduce the cost of the drawdown insurance. Finally, we also add a cancellation feature to this contract.

Zhang et al. (2013) only considered a risky asset modeled by the geometric Brownian motion. However, in recent years, the empirical study of financial data reveals that the distribution of the log-return of stock price exhibits features which cannot be captured by the normal distribution such as heavy tails and asymmetry. With a view to replicating these features more effectively and to reproducing a wide variety of implied volatility skews and smiles, there has been a general shift in the literature to modeling a risky asset with an exponential Lévy process rather than the exponential of a linear Brownian motion; see Kyprianou (2006) and Øksendal and Sulem (2004) for overviews. Therefore looking for a better fitting of the evolution of the stock price process to real data, in this paper we price derivative securities in the market by a general geometric spectrally negative Lévy process. That is, the logarithm of the price of a risky asset in our case will be a process with stationary and independent increments with no positive jumps.

The last contract analyzed in this paper taking into account drawdown and drawup with a cancellation feature is considered for the first time in the literature. Although it is the most complex, it produces very interesting and surprising results. In particular, we discover a new phenomenon for the optimal stopping rule in this contract. In the phenomenon, the investor's stopping rule is also at a first passage time of the drawdown of the log-return process, similarly to the second contract without a drawup contingency. Still, the level of termination is different, taking into account the drawup event.

Our approach is based on the classical fluctuation theory for spectrally negative Lévy processes (related to so-called scale functions) and some new exit identities for reflected Lévy processes. These new formulas identify two-sided exit problems for drawup and drawdown first passage times. A key element of our approach is path analysis and the use of some results of Mijatović and Pistorius (2012). We also heavily use optimal stopping theory. In a market where the underlying dynamics for the stock price process is driven by the exponential of a linear Brownian motion the valuation is transformed into a free boundary problem. However, if we allow jumps to appear in the sample paths of the dynamics of the stock price process, this idea breaks down. To tackle these infinite horizon problems we use the so-called "guess and verify" method. For this method, one guesses what the optimal value function and optimal stopping should be, and then tries to verify that candidate solution is indeed the optimal one by testing it by means of a verification theorem. This means that the value function identified by the guessed stopping rule applied to the log-return price process constructs a smallest, in some sense, discounted supermartingale.

In this paper we also analyze many particular examples and make an extensive numerical analysis showing the dependence of the contract and stopping time on the model's parameters. We mainly focus on the case when the logarithm of the asset price is a linear Brownian motion or drift minus a compound Poisson process (a Cramér–Lundberg risk process).

The paper is organized as follows. In Section 2 we introduce the main definitions, notation, and the main fluctuation identities. We analyze insurance contracts based on drawdown and additional drawup in Sections 3 and 4, respectively. We finish by the numerical analysis in Section 5 and Conclusions in Section 6.

2. Preliminaries

We work on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions. We model the logarithm of the underlying risky asset price $\log S_t$ by a spectrally negative Lévy process X_t, that is, $S_t = \exp\{X_t\}$ is a geometric Lévy process. This means that X_t is a stationary stochastic process with independent increments, right-continuous paths with left limits, and has only negative jumps.

Many identities will be given in terms of so-called scale functions which are defined in the following way. We start by defining the so-called Laplace exponent of X_t:

$$\psi(\phi) = \log \mathbb{E}[e^{\phi X_1}], \tag{1}$$

which is well defined for $\phi \geq 0$ due to the absence of positive jumps. Recall that by the Lévy–Khintchine theorem,

$$\psi(\phi) = \mu\phi + \frac{1}{2}\sigma^2\phi^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-\phi u} - 1 + \phi u \mathbb{1}_{(u<1)}\right)\Pi(\mathrm{d}u), \tag{2}$$

which is analytic for $\mathfrak{Im}(\phi) \leq 0$, where μ and $\sigma \geq 0$ are real and Π is a so-called Lévy measure such that $\int \left(1 \wedge x^2\right)\Pi(\mathrm{d}x) < \infty$. It is easy to observe that ψ is zero at the origin, tends to infinity at infinity and is strictly convex. We denote by $\Phi : [0, \infty) \to [0, \infty)$ the right continuous inverse of ψ so that

$$\Phi(r) = \sup\{\phi > 0 : \psi(\phi) = r\} \quad \text{and}$$
$$\psi(\Phi(r)) = r \quad \text{for all } r \geq 0.$$

For $r \geq 0$ we define a continuous and strictly increasing function $W^{(r)}$ on $[0, \infty)$ with Laplace transform given by

$$\int_0^\infty e^{-\phi u} W^{(r)}(u)\mathrm{d}u = \frac{1}{\psi(\phi) - r} \quad \text{for} \quad \phi > \Phi(r), \tag{3}$$

where ψ is the Laplace exponent of X_t given in (1). $W^{(r)}$ is called the first scale function. The second scale function is related to the first via:

$$Z^{(r)}(u) = 1 + r\int_0^u W^{(r)}(\phi)\mathrm{d}\phi. \tag{4}$$

In this paper we will assume that

$$W^{(r)} \in \mathcal{C}^1(\mathbb{R}_+) \tag{5}$$

for $\mathbb{R}_+ = [0, \infty)$. This assumption is satisfied when the process X_t has a non-trivial Gaussian component, or it is of unbounded variation, or the jumps have a density; see Kyprianou et al. (2013, Lem. 2.4). The scale functions are used in two-sided exit formulas:

$$\mathbb{E}_x\left[e^{-r\tau_a^+}; \tau_a^+ < \tau_0^-\right] = \frac{W^{(r)}(x)}{W^{(r)}(a)}, \tag{6}$$

$$\mathbb{E}_x\left[e^{-r\tau_0^-}; \tau_0^- < \tau_a^+\right] = Z^{(r)}(x) - Z^{(r)}(a)\frac{W^{(r)}(x)}{W^{(r)}(a)}, \tag{7}$$

where $x \leq a$, $r \geq 0$ and

$$\tau_a^+ = \inf\{t \geq 0 : X_t \geq a\}, \qquad \tau_a^- = \inf\{t \geq 0 : X_t \leq a\} \tag{8}$$

are the first passage times and we set $\inf \emptyset = \infty$. We use the notation $\mathbb{E}[\cdot\ \mathbb{1}_{\{A\}}] = \mathbb{E}[\cdot; A]$ for any event A.

Set:

$$\overline{X}_t = \sup_{s\leq t} X_s, \qquad \underline{X}_t = \inf_{s\leq t} X_s.$$

In this paper, we analyze some insurance contracts related to the drawdown and drawup processes of the log-return of the asset price S_t, that is, to the drawdown and drawup processes of X_t. The drawdown and drawup processes are Markov process and are defined as follows. The drawdown is the difference between