Insurance: Mathematics and Economics 79 (2018) 167–183

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Annuitization and asset allocation under exponential utility

Xiaoqing Liang [a], Virginia R. Young [b,*]

[a] School of Sciences, Hebei University of Technology, Tianjin 300401, PR China
[b] Department of Mathematics, University of Michigan, Ann Arbor, MI, 48109, USA

ARTICLE INFO

Article history:
Received July 2017
Received in revised form November 2017
Accepted 17 January 2018

Keywords:
Life annuities
Optimal consumption
Optimal investment
Stochastic control
Free-boundary problem

ABSTRACT

We find the optimal investment, consumption, and annuitization strategies for a retiree who wishes to maximize her expected discounted utility of lifetime consumption. We assume that the retiree's preferences exhibit constant absolute risk aversion (CARA), that is, the retiree's utility function is exponential. The retiree invests in a financial market with one riskless and one risky asset, the so-called Black–Scholes market. Moreover, the retiree may purchase single-premium immediate life annuity income that is payable continuously, and she may purchase this life annuity income at any time and for any amount, subject to the limit of her available wealth.

Because maximizing exponential utility generally does not prevent wealth from dropping below 0, we restrict the investment, consumption, and annuitization strategies so that wealth remains non-negative. We solve the optimization problem via stochastic control and obtain semi-explicit solutions by using the Legendre dual. We prove that the optimal annuitization strategy is a barrier strategy. We also provide some numerical examples to illustrate our results and to analyze their sensitivity to the parameters.

1. Introduction

Asset allocation is an essential topic in actuarial and financial mathematics and in financial economics. Due to the aging of populations around the world and due to the corresponding desire to eliminate longevity risk, we anticipate that life annuities will become one of the most rapidly growing financial products in the next few decades. Milevsky and Young (2007) added life annuities into the asset allocation model of Merton (1969, 1971); see Milevsky and Young (2007) for additional references. Merton (1969) assumed that the investor's preferences exhibit either constant absolute risk aversion (CARA) or constant relative risk aversion (CRRA). Milevsky and Young (2007) analyzed the optimal investment, consumption, and annuitization strategies for a retiree with stochastic time of death and with CRRA preferences. Wang and Young (2012) extended this work by considering commutable life annuities, that is, annuities the retiree can surrender by paying a (proportional) surrender fee.

A utility function that exhibits CARA preferences is exponential in form; thus, utility with CARA preferences is also referred to as exponential utility. Mathematically, exponential utility generally makes the resulting optimization problem tractable. For example, Zeng et al. (2015) investigated a life insurance optimization problem with exponential utility, and Bayraktar and Young (2013) considered an optimization problem for a household of two wage earners with this utility. See, also, Caballero (1990), Svensson and Werner (1993), Llorente et al. (2002), Christensen et al. (2012), and Liu (2004) for portfolio choice problems under exponential utility. However, an essential drawback of exponential utility is that, unless the rate of consumption is constrained to remain non-negative, it might become negative, an unrealistic result; also, (unconstrained) wealth might become negative, an undesirable outcome, especially compared with the related outcome under CRRA utility in which consumption and wealth remain non-negative with probability one. Many authors have recognized this issue (see, for example, Caballero, 1990), but explicit solutions are generally difficult to obtain if non-negative constraints are imposed.[1]

In this paper, we determine the optimal investment, consumption, and annuitization strategies for a retiree with CARA preferences who wishes to maximize her expected discounted utility of lifetime consumption. The retiree invests in a financial market with one riskless and one risky asset. Moreover, the retiree may purchase single-premium immediate life annuity income that is payable continuously, and she may purchase this life annuity income at any time and for any amount, subject to the limit of her

* Corresponding author.
E-mail addresses: liangxiaoqing115@hebut.edu.cn (X. Liang), vryoung@umich.edu (V.R. Young).

[1] Karatzas et al. (1986) studied a general investment and consumption decision problem by imposing a non-negative consumption constraint and a penalty for bankruptcy. Cox and Huang (1989) employed a martingale technique to solve the optimal consumption–investment problem with non-negative constraints on both consumption and final wealth.

https://doi.org/10.1016/j.insmatheco.2018.01.005
0167-6687/

available wealth. In addition, we impose non-negative constraints on both consumption and wealth. By using stochastic control and Legendre dual methods, we obtain semi-explicit solutions for the problem. We show that the optimal annuitization strategy is a barrier strategy. We find that this barrier increases with existing annuity income, decreases with risk aversion, and increases with the price of life annuities.

This paper considers a similar problem as studied in Milevsky and Young (2007) under an "anything anytime" annuity market framework with CRRA (or power) utility. Because of the form of CRRA utility, the value function in Milevsky and Young (2007) is homogeneous with respect to wealth w and annuity income A; thus, the original two-dimensional problem in Milevsky and Young (2007) can be reduced to one dimension. However, homogeneity does not hold for CARA (or exponential) utility; ours is a truly two-dimensional problem and thereby more complicated to solve. Due to the non-negative consumption constraint, our problem separates into three cases, depending on the value of existing annuity income, and we solve each case in turn.[2] In each case, the optimal annuitization strategy is a barrier strategy, as was the optimal strategy under power utility in Milevsky and Young (2007). The barrier is an increasing function of the annuity income A. When A is large, the barrier degenerates to a constant, and when A approaches zero, the barrier reduces to zero.

The remainder of this paper is organized as follows. In Section 2, we introduce the model dynamics, define admissible strategies, and provide a verification theorem. Immediately after the verification theorem, we state our ansatz, which includes the hypothesis that the optimal annuitization strategy is a barrier strategy. In Section 3, we construct a candidate value function based on our ansatz and use the verification theorem to show that the constructed candidate equals the value function. We also state the corresponding optimal investment, consumption, and annuitization strategies in detail. In Section 4, we analyze the properties of the optimal strategies, and we compare the optimal investment and consumption strategies with their correspondences when life annuities are not available in the market. We find that the retiree invests less in the risky asset and consumes more when life annuities are available and when the value of wealth is large. In Section 4, we also illustrate our results via numerical examples. Section 5 concludes the paper.

2. Model formulation

In this section, we first describe the financial and life annuity markets in which the retiree can invest her wealth and purchase life annuities. Then, we formulate the problem of maximizing the retiree's utility of lifetime consumption. Finally, we present a verification theorem that we will use to compute the retiree's value function and corresponding optimal strategies.

2.1. Financial and life annuity markets

We consider an optimization problem for a retiree who can invest in a financial market consisting of one riskless and one risky asset, whose prices evolve according to the dynamics

$$dR_t = r\,R_t\,dt, \quad \text{and} \quad dS_t = \mu\,S_t\,dt + \sigma\,S_t\,dB_t,$$

in which $r > 0$, $\mu > r$, and $\sigma > 0$ are constants. Here, $\{B_t\}$ is a standard Brownian motion with respect to a filtration $\{\mathcal{F}_t\}$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In addition, we let τ_d denote the retiree's future lifetime, which is assumed to be an exponential random variable with parameter λ^s, with $\mathbb{E}[\tau_d] = 1/\lambda^s$. τ_d is defined on the same probability space and is independent of $\{B_t\}$. The parameter λ^s is also called the force of mortality, or hazard rate, which we interpret as representing the retiree's *subjective* belief about her future lifetime (hence, the superscript s).

Moreover, we consider a life annuity market, similar to the one of Milevsky et al. (2006) and Milevsky and Young (2007). The retiree may purchase single-premium immediate life annuity income that is payable continuously, and she may purchase this life annuity income at any time and for any amount, subject to the limit of her wealth. The (single) premium for a life annuity that pays \$1 per year continuously until the retiree dies is given by

$$\bar{a} = \int_0^\infty e^{-rt}\,e^{-\lambda^o t}\,dt = \frac{1}{r+\lambda^o},$$

in which $\lambda^o > 0$ is the (constant) *objective* hazard rate used by the insurance company to price annuities (hence, the superscript o). The insurance company may employ a value of λ^o less than λ^s, and if it does so, then the insurance company effectively loads its annuity price with proportional transaction costs. Whether $\lambda^o < \lambda^s$ or $\lambda^o \geq \lambda^s$, annuities are cheaper than the riskless asset; however, they are not tradable. Once the retiree spends money on a life annuity, she may not commute the annuity.[3]

2.2. Utility of lifetime consumption

We assume that the retiree seeks to maximize her utility of lifetime consumption, without a bequest motive. Let c_t denote her rate of consumption at time t, let π_t denote the amount invested in the risky asset at time t, and let A_t denote the cumulative amount of (immediate) life annuity income purchased at or before time t. Then, in the annuity market, the retiree's wealth evolves according to the following dynamics:

$$\begin{cases} dW_t = [rW_{t-} + (\mu - r)\pi_{t-} - c_{t-} + A_{t-}]\,dt + \sigma\,\pi_{t-}\,dB_t \\ \qquad\quad - \bar{a}dA_t, \quad t \geq 0, \\ W_{0-} = w. \end{cases}$$

The investment, consumption, and annuitization strategies $\{\pi_t, c_t, A_t\}_{t\geq 0}$ are said to be *admissible* if they satisfy the following properties.

(i) The control processes $\{\pi_t\}$, $\{c_t\}$, and $\{A_t\}$ are progressively measurable with respect to the filtration $\{\mathcal{F}_t\}$.
(ii) The investment process $\{\pi_t\}$ satisfies $\int_0^t \pi_s^2\,ds < \infty$ with probability one, for all $t \geq 0$.
(iii) The consumption process $\{c_t\}$ is non-negative and satisfies $\int_0^t c_s\,ds < \infty$ with probability one, for all $t \geq 0$.
(iv) The annuitization process $\{A_t\}$ is non-negative and non-decreasing. $A_{0-} = A \geq 0$ equals pre-existing retirement income.[4]
(v) The associated wealth process $\{W_t\}$ is non-negative with probability one, for all $t \geq 0$.

We write $\mathcal{S}(w, A)$ to denote the class of all admissible strategies when the initial wealth and life annuity income is (w, A).

Following Merton (1969) and others, we assume that the preferences of the retiree exhibit constant absolute risk aversion (CARA), that is,

$$u(c) = -\frac{1}{\gamma}\,e^{-\gamma c}, \tag{2.1}$$

[2] Milevsky and Young (2007) did not need to enforce the constraint that consumption be non-negative; optimal consumption was automatically non-negative for power utility.

[3] Wang and Young (2012) considered a life annuity market in which annuities are commutable.

[4] We often refer to A as initial life annuity income because of the time-homogeneity of our control problem, although it may include other income, such as income from pensions.