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An efficient algorithm for the valuation of a guaranteed annuity option with correlated financial and mortality risks

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ABSTRACT

We introduce a pricing framework for a guaranteed annuity option (GAO) where both the interest and mortality risks are correlated. We assume that the short rate and the force of mortality follow the Cox–Ingersoll–Ross (CIR) and Lee–Carter models, respectively. Employing the change of measure technique, we decompose the pure endowment into the product of the bond price and survival probability, thereby facilitating the evaluation of the annuity expression. With the aid of the dynamics of interest and mortality processes under the forward measure, we construct an algorithm based on comonotonicity theory to estimate the quantiles of survival probability and annuity rate. The comonotonic upper and lower bounds in the convex order are used to approximate the annuity and GAO prices and henceforth avoiding the simulation-within-simulation problem. Numerical illustrations show that our algorithm gives an efficient and practical method to estimate GAO values.

1. Introduction

Recent financial innovations in the market included the creation of many insurance products with option-embedded features such as guaranteed annuity options and equity-linked annuities; see Hardy (2003). These products depend on both mortality and interest rate risks. The previous methodology in evaluating this kind of products affected by these two risks is oversimplified. In the past literature, the interest rate is modelled as a stochastic process and the mortality rate is deemed deterministic; see Ballotta and Haberman (2006, 2003). The underestimation of mortality risk could lead to huge losses for many insurance companies. Majority of research papers do not deal with the correlation between mortality and interest rate risks. It is more desirable to have a valuation setting that allows for the dependence between these two risk factors. In Liu et al. (2014), a valuation pricing framework covers the case of correlated mortality and interest risks, albeit the interest rate model is restricted to Vasiček to obtain analytic pricing solution of a guaranteed annuity option (GAO). Liu et al. (2013), on the other hand, proposed comonotonicity-based method to improve the efficiency of GAO pricing computation.

With the improvement of approaches in modelling mortality risks, more stochastic mortality models with greater flexibility, were put forward; see Cairns et al. (2009), Lee and Carter (1992), and Lin and Liu (2007), amongst others. The pricing of annuity products has become more complicated as the complexity of the mortality model has also increased. We aim to construct a model with greater capability in fitting with the historical data very well, i.e., capturing ably the mortality evolution whilst attaining tractability for ease of implementation. However, building such a model that captures adequately both behavioural properties could be challenging. When a complicated mortality model is adopted, the computational burden is heavy and the "simulation-within-simulation" problem poses a difficulty in the implementation. Our goal is to develop a computationally efficient algorithm to evaluate the GAO price.

GAO valuation with regime-switching but under independent risk factors is put forward in Gao et al. (2015b); the pricing under regime-switching with correlation structure involving Vasiček interest-rate dynamics and mortality rate is given in Gao et al. (2015a); and the setting of GAO capital requirements using moment-based method is shown in Gao et al. (2017). This paper could be viewed as an extension of the framework constructed in Liu et al. (2014) in which their model setting is limited only to Vasiček and Ornstein–Uhlenbeck-based models for the interest and mortality rates, respectively, due to the models' combined mathematical tractability. Nonetheless, we know that both risk processes have more complicated dynamics requiring a combined modelling framework with more capabilities. So, to bring further modelling development, we consider the CIR model for the interest rate process which is mean-reverting and its nonnegative feature provides a realistic description of the evolution of the interest rate.

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Also, the well-studied Lee–Carter mortality model is adopted in this investigation. As pointed out in Lee and Carter (1992), their mortality model performs superbly in fitting the empirical data. However, the price of GAO for this choice of combined interest and mortality models, with correlation structure, does not yield a closed-form pricing solution so that the simulation technique must be used. To aid the price computation, the comonotonicity lower and upper bounds are calculated to give an approximation of the GAO value.

We note that this paper and that of Deelstra et al. (2016) exhibit similarities in GAO pricing. These similarities include the (i) framework of correlated interest and mortality risks, (ii) examination of the influence brought about by the risks' dependence structure on GAO prices, (iii) employment of the change-of-measure technique, and (iv) short-term interest rate governed by the Cox–Ingersoll–Ross (CIR) model.

Nonetheless, this article also has certain features that depict distinctive differences from Deelstra et al. (2016). Such features justify our paper's unique position relative to the current literature, and its contributions by all means complement those in Deelstra et al. (2016). We highlight the differences as follows. (i) We assume the mortality rate evolves according to the Lee–Carter model whilst in Deelstra et al. (2016), both mortality and short rates follow the multi-CIR or Wishart models. The Lee–Carter model arguably performs better in fitting mortality rates as this model was originally created to model mortality risks; in particular, it takes into account both the age and time factors. In contrast, the model in Deelstra et al. (2016) considers only the time factor. (ii) In this paper, the interest and mortality rates are governed by different models vis-à-vis the assumption in Deelstra et al. (2016). Thus, in our case, explicit solutions for the annuity rates and GAO prices are unattainable. But, with the aid of the concept of comonotonicity bounds, we get closed-form pricing approximations for annuity and GAO prices. (iii) The procedure to calculate our survival probability and price estimates is deemed efficient with the general Monte-Carlo simulation method as benchmark. Our numerical results are enhanced further by a systematic analysis that ascertains how sensitive the GAO prices are to the perturbations in various parameter values.

This paper is organised as follows. Section 2 presents the formulation of the pricing framework along with the assumptions of the interest and mortality rate modelling set ups. In Section 3, we describe the change of measure method and determine the dynamics of the interest and mortality rate processes under the forward measure. In Section 4, the comonotonicity bounds are introduced and they are used in turn to evaluate the survival probability, annuity rate, and GAO price. Section 5 provides some numerical examples to illustrate the advantages of our proposed technique. Finally, we give some concluding remarks in Section 6.

2. Valuation framework

2.1. Cox–Ingersoll–Ross (CIR) model

Under a risk-neutral measure Q, the short-term interest rate r_t is governed by the CIR model, i.e., r_t follows the dynamics

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t^1, \tag{1}$$

where a, b and σ are positive constants and W_t^1 is a standard Brownian motion on some filtered probability space $(\Omega, \mathcal{R}_t, \{\mathcal{R}_t\}, Q)$. The square root term in the diffusion coefficient of Eq. (1) together with an appropriate choice of parameters imply that the interest rate is always positive. The respective mean and variance of r_t conditional on $\mathcal{R}_s$ are given by

$$\mathrm{E}[r_t|\mathcal{R}_s] = r_s e^{-a(t-s)} + b(1 - e^{-a(t-s)})$$

and

$$\mathrm{Var}[r_t|\mathcal{R}_s] = r_s\frac{\sigma^2}{a}\left(e^{-a(t-s)} - e^{-2a(t-s)}\right) + \frac{\sigma^2 b}{2a}\left(1 - e^{-a(t-s)}\right)^2.$$

Under the CIR setting, the price of a \$1 time-T zero-coupon bond at time t has an exponential affine representation given by

$$B(t,T) = \mathrm{E}^{Q}\left[e^{-\int_t^T r_u du}\middle|\mathcal{R}_t\right] = e^{-A(t,T)r_t + D(t,T)}, \tag{2}$$

where

$$A(t,T) = \frac{2(e^{(T-t)h} - 1)}{2h + (a+h)(e^{(T-t)h} - 1)},$$

$$D(t,T) = \frac{2ab}{\sigma^2}\log\left(\frac{2he^{(a+h)(T-t)/2}}{2h + (a+h)(e^{(T-t)h} - 1)}\right)$$

and $h = \sqrt{a^2 + 2\sigma^2}$.

2.2. Lee–Carter model

We assume that the force of mortality $\mu_{x,t}$, for a life aged x at time t, follows the Lee–Carter model, which consists of two age-specific factors and a time-varying index. That is,

$$\log\mu_{x,t} = \alpha_x + \beta_x k_t + \epsilon_{x,t} \tag{3}$$

with constraints

$$\sum_t k_t = 0, \quad \sum_x \beta_x = 1,$$

where α_x and β_x are age-specific constants, k_t is a time-varying index, and $\epsilon_{x,t}$ is an error term. In Eq. (3), k_t satisfies the stochastic differential equation

$$dk_t = cdt + \xi dZ_t, \tag{4}$$

where c and ξ are constants ($\xi > 0$) and Z_t is a standard Brownian motion on $(\Omega, \mathcal{M}_t, \{\mathcal{M}_t\}, Q)$.

The Lee–Carter model parameters in Eq. (3) have intuitive interpretations: α_x is the average mortality rate over time that describes the differences between ages; k_t represents the changes of mortality rate over time, which is a stochastic process; β_x explains at which ages mortality rate declines rapidly as influenced by k_t; and $\epsilon_{x,t}$ is a random disturbance.

Given the force of mortality $\mu_{x,t}$, we get the survival probability

$$S_x(t,T) = \exp\left(-\int_t^T u_{x+s,s}ds\right).$$

Since $\mu_{x,t}$ is a stochastic process, the survival probability is also a random variable. To price an insurance product, we need to determine the conditional expectation of the survival probability, which is

$$P_x(t,T) := \mathrm{E}\left[\exp\left(-\int_t^T u_{x+s,s}ds\right)\middle|\mathcal{M}_t\right]. \tag{5}$$

We assume that Z_t is correlated with W_t^1 with dependence structure

$$dZ_t dW_t^1 = \rho dt,$$

where ρ is the correlation coefficient between the interest rate and mortality rate processes. Generating Z_t could be performed using the relation

$$Z_t = \rho W_t^1 + \sqrt{1-\rho^2}W_t^2,$$

where W_t^2 is a standard Brownian motion independent of W_t^1.