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Duality in ruin problems for ordered risk models

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ABSTRACT

On one hand, an ordered dual risk model is considered where the profit arrivals are governed by an order statistic point process (OSPP). First, the ruin time distribution is obtained in terms of Abel–Gontcharov polynomials. Then, by duality, the ruin time distribution is deduced for an insurance model where the claim amounts correspond to the inter-arrival times in an OSPP. On the other hand, an ordered insurance model is considered with an OSPP as claim arrival process. Lefèvre and Picard (2011) determined the finite-time ruin probability in terms of Appell polynomials. Duality is used to derive the ruin probability in a dual model where the profit sizes correspond to the inter-arrival times of an OSPP.

1. Introduction

Dual risk models describe the wealth of a company for which the operational cost is deterministic and the profits occur stochastically. Their appellation comes from the duality with insurance (or primal) risk models for which the premium income is deterministic and the claims arrive stochastically. There is an extensive literature on insurance risk models. Dual risk models have received an increasing interest in recent years.

Dual risk model. A company holds an initial capital $v > 0$ and faces running costs at a constant rate $a > 0$. The company makes profits over time that form a sequence $\{Y_i, i \geq 0\}$ of i.i.d. non-negative random variables. These profits occur according to a counting process $\{M(t), t \geq 0\}$, independently of the Y_i. The associated wealth process $\{W(t), t \geq 0\}$ is given by

$$W(t) = v - at + \sum_{i=1}^{M(t)} Y_i, \quad t \geq 0. \tag{1}$$

The ruin time, σ_v, is the first instant at which the wealth process falls at the level 0, i.e.

$$\sigma_v = \inf\{t \geq 0 : W(t) = (\leq) 0\}. \tag{2}$$

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The dual risk model is considered e.g. in the books by Cramér (1955), Seal (1967), Takács (1967), Grandell (1991) and Asmussen and Albrecher (2010). The model is suitable for risky business sectors such as oil prospection, pharmaceutical research or new technology development. So, Bayraktar and Egami (2008) use it to describe the financial reserves of venture capital funds, and Bertail et al. (2008) to model the exposure to a given food contaminant. Another application is in life insurance when a company pays annuities on a regular basis and receives a part of the reserves at each policyholder death.

Many of the works on dual models focus on optimal dividend problems. We refer e.g. to Avanzi et al. (2007), Gerber and Smith (2008), Albrecher et al. (2008), Dai et al. (2010), Cheung (2012), Afonso et al. (2013), and Bergel et al. (2014). First-passage problems and ruin time are also much studied; see e.g. Landriault and Sendova (2011), Mazza and Rullière (2004), Zhu and Yang (2008), Yang and Sendova (2014) and Dimitrova et al. (2015). There exists here a close and important connection with queueing models; see e.g. Frostig (2004), Badescu et al. (2005) and Frostig and Keren-Pinhasik (2017).

The classical dual model is the compound Poisson case where the counting process $\{M(t), t \geq 0\}$ in (1) is a Poisson process. The Sparre-Andersen case where $\{M(t), t \geq 0\}$ is a renewal process and other extensions like the Markov-modulated case have been investigated to a certain extent.

In this paper, we first examine a dual risk model where $\{M(t), t \geq 0\}$ is an order statistic point process (OSPP). Such a model is named ordered dual in the sequel. The class of OSPP was characterized by Puri (1982), further to earlier partial results. The key property of an OSPP is that conditionally on the number of profit arrivals up to time $t \geq 0$, the jump times are distributed as the order statistics for a random sample drawn from some continuous distribution with support $(0, t)$. The OSPP class covers the (mixed) Poisson process, the linear birth process with immigration and the linear death counting process. This class of counting processes was proposed to model claim frequencies in insurance by Lefèvre and Picard (2011), after the pioneering works of De Vylder and Goovaerts (1999, 2000). It was used later by Sendova and Zitikis (2012), Lefèvre and Picard (2014, 2015) and Dimitrova et al. (2016).

For that dual model, our purpose is to derive an explicit formula for the distribution of the ruin time σ_v. To this end, we use the representation of the joint distribution of the order statistics from a uniform distribution through the so-called Abel–Gontcharov polynomials. This family of polynomials is little known and related to the more standard Appell polynomials. A review on both polynomial families is provided in Lefèvre and Picard (2015), with applications in risk modelling.

Insurance risk model. An insurance company has an initial capital $u \geq 0$ and receives premiums at a constant rate $c > 0$. The company covers claim amounts over time that form a sequence $\{X_i, i \geq 0\}$ of i.i.d. non-negative random variables. These claims occur according to a counting process $\{N(t), t \geq 0\}$, independently of the X_i. The associated reserve process $\{R(t), t \geq 0\}$ is given by

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad \geq 0. \tag{3}$$

The ruin time, τ_u, is the first instant at which the reserve process becomes negative, i.e.

$$\tau_u = \inf\{t \geq 0 : R(t) < 0\}. \tag{4}$$

Much research is devoted to insurance risk models, especially for ruin related problems. The reader is referred e.g. to the books of Asmussen and Albrecher (2010) and Dickson (2005). The traditional case is the compound Poisson model where the counting process $\{N(t), t \geq 0\}$ in (3) is a Poisson process. The Sparre-Andersen model supposes that $\{N(t), t \geq 0\}$ is a renewal process. A number of generalizations and variants of these models have been considered. This is the case e.g. of the mixed Poisson model (see Grandell, 1997).

It is well-known that to the insurance risk model (3) is associated a dual risk model (1) whose characteristics are inverted in a precise sense. Specifically, the profits in the dual model correspond to the inter-arrival times in the insurance model while the inter-arrival times in the dual model correspond to the claim sizes in the insurance model, and the cost rate in the dual model is the inverse of the premium rate in the insurance model. This duality is pointed out and exploited in various works in insurance. Let us mention e.g. the recent papers by Shi and Landriault (2013), Mazza and Rullière (2004), Borovkov and Dickson (2008) and Dimitrova et al. (2015). A similar duality property is used with Lévy processes in finance.

This link between the primal and dual models can provide a simple approach for tackling ruin problems. Mazza and Rullière (2004) start with the compound Poisson dual model, a recursive formula for the finite-time ruin probability is derived (similar to existing recursive formula for the finite-time ruin probability in the insurance risk model, see e.g. Picard and Lefèvre, 1997, Rullière and Loisel, 2004 and Lefèvre and Loisel, 2009) and then pass to the insurance model with exponential claim amounts. Dimitrova et al. (2015) apply results for the insurance model to obtain the ruin probability in the corresponding dual model.

In the same vein, we will make here a round trip between the dual and insurance models. As announced above, we first derive a formula for the distribution of the ruin time σ_v in an ordered dual risk model. From this formula, we then deduce the distribution of the ruin time τ_u in the Sparre-Andersen insurance model where the claim arrivals are governed by a renewal process and the claim amounts are distributed as the inter-arrival times in an OSPP. As a special case, we recover a result obtained by Borovkov and Dickson (2008) for the case of i.i.d. exponential claim amounts.

Our second journey is from the primal to the dual and concerns now the finite-time ruin probability. We start with an ordered insurance model where the claim arrivals are described by an OSPP. For this model, Lefèvre and Picard (2011, 2014) derived a formula for the ruin probability in terms now of Appell polynomials. By duality, we can then obtain the finite-time ruin probability in the associated dual risk model where the profit arrivals are governed by a renewal process and the profit sizes are distributed as the inter-arrival times in an OSPP.

It is worth underlining that the approach by duality enables us to deal with ruin problems in primal or dual risk models of renewal type which allow for some dependence between the claim or profit sizes. Models with dependent claims or profits are usually difficult to study. The present method, rather simple, relies on a preliminary study of the ruin in the associated dual and primal models. Such a study is possible when the latter models are ordered, i.e. the profits or claims arrive according to an OSPP.

Summary. The paper is organized as follows. Section 2 gives an overview of the order statistic point processes. In Section 3, we obtain the ruin time distribution in the ordered dual model when the profits arrivals are governed by an OSPP. In Section 4, we deduce the ruin time distribution in a Sparre-Andersen insurance model where the claim amounts correspond to the inter-arrival times of an OSPP. In Section 5, we obtain the finite-time ruin probability in the associated dual model where the profit sizes correspond to the inter-arrival times of an OSPP.

2. Order statistic property

The Poisson process is the traditional model for counting events that arise randomly in the course of time. Of simple construction, it has also many desirable properties. In particular, it belongs to the class of order statistic point processes.

Definition 2.1. A point process $\{N(t), t \geq 0\}$, with $N(0) = 0$, is an OSPP if for every $n \geq 1$, provided $\mathbb{P}[N(t) = n] > 0$, then conditioned upon $[N(t) = n]$, the successive jump times $(T_1, T_2, \ldots, T_n)$ are distributed as the order statistics $[U_{1:n}(t), \ldots, U_{n:n}(t)]$ of a sample of n i.i.d. random variables on $[0, t]$, distributed as a variable $\mathcal{U}(t)$ of distribution function $\mathbb{P}[\mathcal{U}(t) \leq s] = F_t(s)$, $0 \leq s \leq t$.

De Vylder and Goovaerts (1999, 2000) introduced a risk model, named homogeneous, that generalizes the classical Cramér–Lundberg risk model. When defined on an infinite horizon, this model supposes that the claim arrival process satisfies the above order statistic property where $\mathcal{U}(t)$, $t > 0$, is uniform on $(0, t)$ (as it is for the Poisson process). Their research was made independently of the existing literature on OSPP.

More recently, Lefèvre and Picard (2011) introduced an insurance risk model in which claim arrivals are described by an OSPP. This paper was continued in Lefèvre and Picard (2014, 2015). The purpose is to determine the finite-time ruin probability in such a model. We also mention the related works by Sendova and Zitikis (2012), Dimitrova et al. (2016) and Goffard (2017). Outside of the insurance context, Goffard and Lefèvre (2017) studied the first-crossing problem of an OSSP through general boundaries.

A complete representation of the class of OSPP was obtained by Puri (1982), following on earlier works. A key result is recalled below.