Insurance: Mathematics and Economics 78 (2018) 136–156

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Approximation of ruin probabilities via Erlangized scale mixtures

Oscar Peralta [a,*], Leonardo Rojas-Nandayapa [b], Wangyue Xie [c], Hui Yao [c]

[a] Department of Applied Mathematics and Computer Science, Technical University of Denmark, Denmark
[b] Mathematical Sciences, University of Liverpool, UK
[c] School of Mathematics and Physics, The University of Queensland, Australia

ARTICLE INFO

Article history:
Received December 2016
Received in revised form December 2017
Accepted 11 December 2017
Available online 20 December 2017

Keywords:
Phase-type
Erlang
Scale mixtures
Infinite mixtures
Heavy-tailed
Ruin probability

ABSTRACT

In this paper, we extend an existing scheme for numerically calculating the probability of ruin of a classical Cramér–Lundberg reserve process having absolutely continuous but otherwise general claim size distributions. We employ a dense class of distributions that we denominate *Erlangized scale mixtures* (ESM) that correspond to nonnegative and absolutely continuous distributions which can be written as a Mellin–Stieltjes convolution $\Pi \star G$ of a nonnegative distribution Π with an Erlang distribution G. A distinctive feature of such a class is that it contains heavy-tailed distributions.

We suggest a simple methodology for constructing a sequence of distributions having the form $\Pi \star G$ with the purpose of approximating the integrated tail distribution of the claim sizes. Then we adapt a recent result which delivers an explicit expression for the probability of ruin in the case that the claim size distribution is modeled as an Erlangized scale mixture. We provide simplified expressions for the approximation of the probability of ruin and construct explicit bounds for the error of approximation. We complement our results with a classical example where the claim sizes are heavy-tailed.

1. Introduction

In this paper, we propose a new numerical scheme for the approximation of ruin probabilities in the classical compound Poisson risk model—also known as Cramér–Lundberg risk model (cf. Asmussen and Albrecher, 2010). In such a risk model, the surplus process is modeled as a compound Poisson process with negative linear drift and a nonnegative jump distribution F, the later corresponding to the claim size distribution. The ruin probability within infinite horizon and initial capital u, denoted $\psi(u)$, is the probability that the supremum of the surplus process is larger than u. The Pollaczek–Khinchine formula (see Eq. (3.1)) provides the exact value of $\psi(u)$, though it can be explicitly computed in very few cases. Such a formula is a functional of $\widehat{F}$, the integrated tail distribution of F; from here on, we will use $\psi_{\widehat{F}}(u)$ instead of $\psi(u)$ to denote this dependence. A useful fact is that the Pollaczek–Khinchine formula can be naturally extended in order to define $\psi_G(u)$ even if G does not correspond to an integrated tail distribution; in this case, $\psi_G(\cdot)$ corresponds to the survival probability of certain terminating renewal process.

The approach advocated in this paper is to approximate the integrated claim size distribution $\widehat{F}$ by using the family of *phase-type scale mixture distributions* introduced in Bladt et al. (2015), but we also consider the more common approach of approximating the claim size distribution F. The family of phase-type scale mixture distributions is dense within the class of nonnegative distributions, and it is formed by distributions which can be expressed as a Mellin–Stieltjes convolution, denoted $\Pi \star G$, of an arbitrary nonnegative distribution Π and a phase-type distribution G (cf. Bingham et al., 1987). The Mellin–Stieltjes convolution corresponds to the distribution of the product between two independent random variables having distributions Π and G, respectively.

In particular, if Π is a nonnegative discrete distribution and $\Pi \star G$ is itself the integrated tail of a phase-type scale mixture distribution, then an explicit computable formula for the ruin probability $\psi_{\Pi\star G}(u)$ of the Cramér–Lundberg process with claims having integrated tail distribution $\Pi \star G$ is given in Bladt et al. (2015). Hence, it is plausible that if $\Pi \star G$ is *close enough* to the integrated tail distribution $\widehat{F}$ of the claim sizes, then we can use $\psi_{\Pi\star G}(u)$ as an approximation for $\psi_{\widehat{F}}(u)$, the ruin probability of a Cramér–Lundberg process having claim size distribution F. One of the key features of the class of phase-type scale mixtures is that if Π has unbounded support, then $\Pi \star G$ is a heavy-tailed distribution (Rojas-Nandayapa and Xie, 2017; Su and Chen, 2006; Tang, 2008), confirming the hypothesis that the class of phase-type scale mixtures is more appropriate for approximating tail-dependent quantities involving heavy-tailed distributions. In contrast, the class of classical phase-type distributions is light-tailed and approximations derived from this approach may be

* Corresponding author.
E-mail addresses: osgu@dtu.dk (O. Peralta), leorojas@liverpool.ac.uk (L. Rojas-Nandayapa), w.xie1@uq.edu.au (W. Xie), h.yao@uq.edu.au (H. Yao).

https://doi.org/10.1016/j.insmatheco.2017.12.005
0167-6687/

inaccurate in the tails (see also Vatamidou et al., 2014 for an extended discussion).

Our contribution is to propose a systematic methodology to approximate any absolutely continuous integrated tail distribution $\widehat{F}$ using a particular subclass of phase-type scale mixtures called *Erlangized scale mixtures* (ESM). The proposed approximation is particularly precise in the tails and the number of parameters remains controlled. Our construction requires a sequence $\{\Pi_m : m \in \mathbb{N}\}$ of nonnegative discrete distributions having the property $\Pi_m \to \widehat{F}$ (often taken as a discretization of the target distribution over some countable subset of the support of $\widehat{F}$), and a sequence of Erlang distributions with equal shape and rate parameters, denoted $G_m \sim \text{Erlang}(\xi(m), \xi(m))$. If the sequence $\xi(m) \in \mathbb{N}$ is unbounded, then $\Pi_m \star G_m \to \widehat{F}$. We adapt the results in Bladt et al. (2015) to compute $\psi_{\Pi_m \star G_m}(u)$, and use this as an approximation of the ruin probability of interest.

To assess the quality of $\psi_{\Pi_m \star G_m}(u)$ as an approximation of $\psi_{\widehat{F}}(u)$, we identify two sources of theoretical error. The first source of error comes from approximating $\widehat{F}$ via Π_m, so we refer to this as the *discretization error*. The second source of error is due to the Mellin–Stieltjes convolution with G_m, so this will be called the *Erlangization error*. The two errors are closely intertwined so it is difficult to make a precise assessment of the effect of each of them in the general approximation. Instead, we use the triangle inequality to separate these as follows

$$\underbrace{\left|\psi_{\widehat{F}}(u) - \psi_{\Pi_m \star G_m}(u)\right|}_{\text{Approximation error}} \leq \underbrace{\left|\psi_{\widehat{F}}(u) - \psi_{\widehat{F} \star G_m}(u)\right|}_{\text{Erlangization error}} + \underbrace{\left|\psi_{\widehat{F} \star G_m}(u) - \psi_{\Pi_m \star G_m}(u)\right|}_{\text{Discretization error}}.$$

Therefore, the error of approximating $\psi_{\widehat{F}}(u)$ with $\psi_{\Pi_m \star G_m}(u)$ can be bounded from above with the aggregation of the Erlangization error and the discretization error. In our developments below, we construct explicit tight bounds for each source of error.

We remark that the general formula for $\psi_{\Pi \star G}(u)$ in Bladt et al. (2015) is computational intensive and can be difficult or even infeasible to implement since it is given as an infinite series with terms involving products of finite dimensional matrices. We show that for our particular model, $\psi_{\Pi \star G_m}(u)$ can be simplified down to a manageable formula involving the probability density function (pdf) and cumulative distribution function (cdf) of the negative binomial distribution instead of computationally expensive matrix operations. In practice, the infinite series can be computed only up to a finite number of terms, but as we will show, this numerical error can be controlled by selecting an appropriate distribution Π. This truncated approximation of $\psi_{\Pi \star G}(u)$ will be denoted $\widetilde{\psi}_{\Pi \star G}(u)$. We provide explicit bounds for the numerical error induced by truncating the infinite series.

All things considered, we contribute to the existing literature for computing ruin probabilities for the classical Cramér–Lundberg model by proposing a new practical numerical scheme. Our method, coupled with the bounds for the error of approximation, provides an attractive alternative for computing ruin probabilities based on a simple, yet effective idea.

The approach described above is a further extension to the use of phase-type distributions for approximating general claim size distributions (cf. Asmussen, 2003; Latouche and Ramaswami, 1999; Neuts, 1975). Several attempts to approximate the probability of ruin for Cramér–Lundberg model have been made before (see Vatamidou et al., 2013 and references therein). A recent and similar approach can be found in Santana et al. (2016) which uses discretization and Erlangizations argument as its backbone. We emphasize here that we address the problem of finding the probability of ruin differently. Firstly, we propose to directly approximate the integrated tail distribution instead of the claim size distribution. This will yield far more accurate approximations of the probability of ruin. Secondly, since we investigate the Erlangization and the discretization errors separately, we are able to provide tight error bounds for our approximation method. This will prove to be helpful in challenging examples such as the one presented here: the heavy-traffic Cramér–Lundberg model with Pareto distributed claims. Lastly, each approximation of ours is based on a mixture of Erlang distributions of fixed order, while the approach in Santana et al. (2016) is based on a mixture of Erlang distributions of increasing order. By keeping the order of the Erlang distribution in the mixture fixed, we can smartly allocate more computational resources in the discretization part, yielding an overall better approximation. More importantly, we find the use of ESM more natural because increasing the order of the Erlang distributions in the mixture translates in having different levels of accuracy of Erlangization at different points. The choice of having sharper Erlangization in the tail of the distribution than in the body seems arbitrary and is actually not useful tail-wise because the tail behavior of $\Pi * G_m$ is the same for each $\xi(m) \geq 1$.

The rest of the paper is organized as follows. Section 2 provides an overview of the main concepts and methods. In Section 3, we present the methodology for constructing a sequence of distributions of the form $\Pi_m \star G_m$ approximating a nonnegative continuous distribution. Based on the results of Bladt et al. (2015), we introduce two simplified infinite series representations of the ruin probability. In Section 4, we construct the bounds for the error of each approximation. In Section 5, we provide a bound for the numerical errors of approximation induced by truncating the infinite series representation. A numerical example illustrating the sharpness of our results is given in Section 6.2. Some conclusions are drawn in Section 7.

2. Preliminaries

In this section, we provide a summary of basic concepts needed for this paper. In Section 2.1 we introduce the family of classical phase-type (PH) distributions and their extensions to phase-type scale mixtures and infinite dimensional phase-type (IDPH) distributions. We will refer to the former class of distributions as *classical* in order to make a clear distinction from the two later classes of distributions. In Section 2.2, we introduce a systematic method for approximating nonnegative distributions within the class of phase-type scale mixtures; such a method will be called approximation via *Erlangized scale mixtures* (ESM). The resulting approximating distribution will be considerably tractable due to the special structure of the Erlang distribution.

2.1. Phase-type scale mixtures

A phase-type (PH) distribution corresponds to the distribution of the absorption time of a Markov jump process $\{X_t\}_{t \geq 0}$ with a finite state space $E = \{0, 1, 2, \ldots, p\}$. The states $\{1, 2, \ldots, p\}$ are transient while the state 0 is an absorbing state. Phase-type distributions are characterized by a p-dimensional row vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)$, corresponding to the initial probabilities of each of the transient states of the Markov jump process, and an intensity matrix

$$\mathbf{Q} = \begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{\lambda} & \boldsymbol{\Lambda} \end{pmatrix}.$$

The subintensity matrix $\boldsymbol{\Lambda}$ corresponds to the transition rates among the transient states while the column vector $\boldsymbol{\lambda}$ corresponds to the exit probabilities to the absorption state. Since $\boldsymbol{\lambda} = -\boldsymbol{\Lambda}\boldsymbol{e}$, where $\boldsymbol{e}$ is a column vector with all elements to be 1, then the pair $(\boldsymbol{\beta}, \boldsymbol{\Lambda})$ completely characterizes the absorption distribution; the notation $\text{PH}(\boldsymbol{\beta}, \boldsymbol{\Lambda})$ is reserved for such a distribution. The cdf,