Insurance: Mathematics and Economics 79 (2018) 243–246

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

On existence and uniqueness of the principle of equivalent utility under Cumulative Prospect Theory

J. Chudziak

Faculty of Mathematics and Natural Sciences, University of Rzeszów, Pigonia 1, 35-959 Rzeszów, Poland

ARTICLE INFO

Article history:
Received June 2017
Received in revised form February 2018
Accepted 9 February 2018
Available online 14 February 2018

Keywords:
Principle of equivalent utility
Existence
Uniqueness
Cumulative Prospect Theory
Choquet integral

ABSTRACT

We establish a necessary and sufficient condition for the existence and uniqueness of the principle of equivalent utility under Cumulative Prospect Theory.

1. Introduction

According to each insurance contract, the insured pays a premium for a protection from an insurance company against an insurable risk. There are several methods of determining the premium. One of them is *the principle of equivalent utility*, belonging to the so-called economic methods of insurance contracts pricing. The principle, proposed by Bühlmann (1970), involves the notion of a utility function and postulates a fairness in terms of utility. This means that the premium is defined in such a way that the insurance company is indifferent between rejecting the contract and entering into it. Let us recall that, if $w \in \mathbb{R}$ is an initial wealth of the insurance company and $u : \mathbb{R} \to \mathbb{R}$ is its continuous and strictly increasing utility function, then the premium of equivalent utility for a risk X, represented by a non-negative bounded random variable on a given probability space, is defined as a unique solution $H(X)$ of the equation

$$E[u(w + H(X) - X)] = u(w). \tag{1}$$

Eq. (1) defines a functional on a family of all risks, called *the principle of equivalent utility*. In the case $w = 0$ the functional is usually referred to as *the zero utility principle*. Several results concerning these principles under Expected Utility Theory can be found e.g. in Bowers et al. (1986), Bühlmann (1970), Gerber (1979) and Rolski et al. (1999).

Recently, the principle of equivalent utility has been extended onto various models of decision making under risk. Heilpern (2003) proposed and investigated the principle under rank-dependent utility model. Kałuszka and Krzeszowiec (2012) introduced the principle of equivalent utility based on Cumulative Prospect Theory developed by Tversky and Kahneman (1992). Some properties of the principle in the latter setting have been studied in Kałuszka and Krzeszowiec (2012, 2013).

It has been noted in Kałuszka and Krzeszowiec (2012) that, in order to get the existence and uniqueness of the principle, it is necessary to impose certain conditions (continuity, strict monotonicity) on a value function. However, it turns out that, in general, the uniqueness of the principle depends not only on the properties of the value function, but also on the relation between the probability distortion functions for gains and losses. The aim of this paper is to establish a necessary and sufficient condition for the existence and uniqueness of the principle of equivalent utility under Cumulative Prospect Theory.

The paper is organized as follows. In Section 2 we recall the definition of the principle of equivalent utility under Cumulative Prospect Theory. The main results are presented in Section 3. Section 4 contains a conclusion. The proofs of auxiliary results are included in the Appendix.

2. Principle of equivalent utility under Cumulative Prospect Theory

As we have already pointed out, the principle of equivalent utility under Cumulative Prospect Theory has been introduced by Kałuszka and Krzeszowiec (2012). The principle is based on the notion of the generalized Choquet integral. In order to recall this notion, assume that $\mathcal{X}$ is a family of all bounded random variables

E-mail address: chudziak@ur.edu.pl.

https://doi.org/10.1016/j.insmatheco.2018.02.001
0167-6687/

on a given probability space. Let $g : [0, 1] \to [0, 1]$ be *a probability distortion function*, that is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$. The Choquet integral of $X \in \mathcal{X}$ with respect to g is defined as follows

$$E_g[X] = \int_{-\infty}^{0} (g(P(X > t)) - 1)\, dt + \int_{0}^{\infty} g(P(X > t))\, dt. \tag{2}$$

The *generalized Choquet integral* related to the distortion functions g (for gains) and h (for losses) is given by

$$E_{gh}[X] = E_g[\max\{X, 0\}] - E_h[\max\{-X, 0\}] \quad \text{for} \quad X \in \mathcal{X}. \tag{3}$$

Under Cumulative Prospect Theory, the premium of equivalent utility for a risk, represented by a non-negative bounded random variable X, is defined through the equation

$$E_{gh}[u(w + H(X) - X)] = u(w), \tag{4}$$

where $w \in \mathbb{R}$ is a difference between an initial wealth and a reference point of an insurance company and $u : \mathbb{R} \to \mathbb{R}$ is its value function. It is remarkable that if g and h are *conjugated*, that is if

$$h(p) = 1 - g(1 - p) \quad \text{for} \quad p \in [0, 1],$$

then $E_{gh}[X] = E_g[X]$ for $X \in \mathcal{X}$ and so Eq. (4) becomes

$$E_g[u(w + H(X) - X)] = u(w). \tag{5}$$

Eq. (5) defines the principle of equivalent utility under rank-dependent utility. Several properties of that principle have been proved by Heilpern (2003).

3. Main results

In what follows, we assume that $\mathcal{X}_+$ is a family of all non-negative bounded random variables on a given non-atomic probability space. Furthermore, $w \in \mathbb{R}$ is a difference between an initial wealth and a reference point of the insurance company, $u : \mathbb{R} \to \mathbb{R}$ is its continuous and strictly increasing value function, with $u(0) = 0$, and $g, h : [0, 1] \to [0, 1]$ are continuous probability distortion functions for gains and losses, respectively. For $X \in \mathcal{X}_+$, we put $m_X := \operatorname{ess\,inf} X$ and $M_X := \operatorname{ess\,sup} X$. Furthermore, for every $X \in \mathcal{X}_+$, the functions $G_X, H_X : \mathbb{R} \to [0, 1]$ and $\phi_X : \mathbb{R} \to \mathbb{R}$ are given by

$$G_X(t) = g(P(X < t)) \quad \text{for} \quad t \in \mathbb{R}, \tag{6}$$

$$H_X(t) = h(P(X > t)) \quad \text{for} \quad t \in \mathbb{R} \tag{7}$$

and

$$\phi_X(s) = E_{gh}[u(w + s - X)] \quad \text{for} \quad s \in \mathbb{R}, \tag{8}$$

respectively. Note that, as g and h are continuous and non-decreasing, for every $X \in \mathcal{X}_+$, G_X is left-continuous and non-decreasing, while H_X is right-continuous and non-increasing.

The next theorem is the main result of the paper.

Theorem 3.1.

(*a*) *If* $w \neq 0$ *then for every* $X \in \mathcal{X}_+$ *there is a unique real number* $H(X)$ *such that* (4) *is valid.*

(*b*) *If* $w = 0$ *then the following statements are equivalent:*

(i) *for every* $X \in \mathcal{X}_+$ *there is a unique real number* $H(X)$ *such that* (4) *is valid;*

(ii)

$$g(1 - p) + h(p) > 0 \quad \text{for} \quad p \in [0, 1]. \tag{9}$$

The following three auxiliary results play an essential role in the proof of Theorem 3.1.

Lemma 3.1. *For every* $X \in \mathcal{X}_+$, *we have*

$$\phi_X(s) = \alpha_X(s) + \beta_X(s) \quad \text{for} \quad s \in \mathbb{R}, \tag{10}$$

where $\alpha_X, \beta_X : \mathbb{R} \to \mathbb{R}$ *are given by*

$$\alpha_X(s) = \int_0^{u(w+s-m_X)} G_X(w + s - u^{-1}(t))\, dt \quad \text{for} \quad s \in \mathbb{R} \tag{11}$$

and

$$\beta_X(s) = -\int_{u(w+s-M_X)}^{0} H_X(w + s - u^{-1}(t))\, dt \quad \text{for} \quad s \in \mathbb{R}. \tag{12}$$

Lemma 3.2. *For every* $X \in \mathcal{X}_+$, *the function* ϕ_X *is continuous.*

Lemma 3.3. *Let* $X \in \mathcal{X}_+$. *Suppose that*

$$\phi_X(s_1) = \phi_X(s_2) =: d \quad \text{for some} \quad s_1, s_2 \in \mathbb{R},\ s_1 < s_2. \tag{13}$$

Then $d = 0$ *and there exists* $p_0 \in (0, 1)$ *such that*

$$g(1 - p_0) + h(p_0) = 0. \tag{14}$$

Proof of Theorem 3.1. Let $X \in \mathcal{X}_+$. Since the generalized Choquet integral is monotone (cf. e.g. Lemma 1 in Kałuszka and Krzeszowiec (2012)), we have

$$E_{gh}[u(w + m_X - X)] \le E_{gh}[u(w)] \le E_{gh}[u(w + M_X - X)].$$

Thus, as $E_{gh}[u(w)] = u(w)$, in view of (8), we get

$$\phi_X(m_X) \le u(w) \le \phi_X(M_X). \tag{15}$$

Moreover, according to Lemma 3.2, ϕ_X is continuous. Hence, there is $s_0 \in \mathbb{R}$ such that $\phi_X(s_0) = u(w)$. Furthermore, applying Lemma 3.3, we conclude that if $w \neq 0$ or (9) holds then such a s_0 is unique. This proves assertion (*a*) and shows that in the case where $w = 0$, we have $(ii) \Longrightarrow (i)$.

Now, suppose that $w = 0$ and (*ii*) does not hold. Then (14) is valid for some $p_0 \in (0, 1)$. Let $x \in (0, \infty)$. Since the probability space is non-atomic, there is a random variable X taking the values 0 and x with probabilities $1 - p_0$ and p_0, respectively. Thus, making use of (2)–(3), we obtain

$$E_{gh}[u(s - X)] = g(1 - p_0)u(s) + h(p_0)u(s - x) = 0 \quad \text{for} \quad s \in [0, x].$$

This means that (*i*) does not hold. Therefore, we have proved that $(i) \Longrightarrow (ii)$, which completes the proof of assertion (*b*).

Remark 1. If g and h are conjugated then $g(1 - p) + h(p) = 1$ for $p \in [0, 1]$. Hence, applying Theorem 3.1, we obtain that for every $X \in \mathcal{X}_+$ there exists a unique real number $H(X)$ such that (5) holds. Therefore, for every $w \in \mathbb{R}$ and every continuous probability distortion function, the principle of equivalent utility under rank-dependent utility, introduced by Heilpern (2003), is uniquely defined.

4. Conclusion

The principle of equivalent utility under Cumulative Prospect Theory has been introduced by Kałuszka and Krzeszowiec (2012). They have pointed out that, in order to get the existence and uniqueness of the principle, it is necessary to impose some additional conditions (continuity, strict monotonicity) on a value function. However, it turns out that, in general, the uniqueness