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Phase recovery from fringe patterns using the continuous wavelet transform

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Abstract

Interferometry is well established as an optical technique in which a measurand is encoded as the phase of a periodically varying intensity pattern. In view of the inherent accuracy of interferometry, many methods have been developed to retrieve the phase from images of the fringe pattern. Our focus in this paper is one such technique—the continuous wavelet transform (CWT). We begin by reviewing the CWT and the space–spatial–frequency localisation properties of wavelets. We show that a path which follows the maximum modulus of the CWT (the wavelet ridge) gives the instantaneous fringe frequency as a function of spatial displacement. The phase is automatically and trivially obtained, without discontinuities, by integration. Examples of practical wavelets are given and algorithms to isolate the wavelet ridge reviewed.

Keywords: Continuous wavelet transform; Wavelets; Fringe patterns; Phase evaluation

1. Introduction

Whole-field optical measurement techniques that encode a measurand as the phase of a periodically varying intensity pattern constitute a powerful set of metrology tools. In general, the two-dimensional fringe pattern $I(x, y)$ so generated has the form

$$I(x, y) = I_0(x, y)[1 + V(x, y)\cos\phi(x, y)], \tag{1}$$

where I_0 is the background intensity and V the fringe visibility. We note that non-sinusoidal fringes, such as those generated by multi-beam interferometers and Moiré methods, are readily represented by their harmonic expansions. A great many techniques have been developed to extract the phase ϕ from an image, or sequence of images, of the fringe pattern and a review of many of these may be found in [1]. Perhaps, the best known spatial technique for extracting phase distributions from a single image is the Fourier transform method [2,3]. Except in certain special cases [4], this method requires the fringes to be modulated onto a spatial carrier. With this proviso, however, the Fourier transform method converts the fringe pattern into a complex-valued analytic signal in the spatial domain through filtering and down conversion in the frequency domain. The phase ϕ is then obtained, modulo 2π, from the arctangent of the imaginary part of the analytic signal divided by the real part. The resulting phase map has discontinuities and requires unwrapping. In principle, this is a straightforward process. In practice, however, the phase unwrapping may be difficult to automate and this has consequently led to the development of numerous phase unwrapping algorithms [1,5,6].

The continuous wavelet transform (CWT), on the other hand, provides a spatial phase recovery method that has several advantages compared to Fourier-based methods. It may be applied to fringes without [7], or with [8], a spatial carrier. For the former, it is necessary to have one additional image, with an arbitrary phase step, in order to resolve the sign ambiguity associated with the slope. In either case, the phase can be automatically recovered, without discontinuities, by integrating the instantaneous fringe frequency obtained by noting the points in transform space for which the modulus of the CWT is a maximum. Finally, the method provides inherent filtering of noisy

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fringes [7] and allows a de-noised version of the fringe to be reconstructed from the wavelet coefficients [10,11].

In Section 2, we introduce the continuous wavelet transform and discuss briefly some aspects concerning its numerical evaluation before providing a few examples of practical wavelets. Section 3 provides a discussion of the space–frequency localisation properties of wavelets, permitting the two-dimensional plot of the modulus of the CWT—the scalogram—to be viewed as the local energy density of the signal. A path which follows the maximum modulus of the CWT is termed the wavelet ridge, and in Section 4 we show that points on the ridge give the instantaneous fringe frequency. We conclude, in Section 5, by reviewing methods to isolate the wavelet ridge and subsequently retrieve the phase.

2. The continuous wavelet transform

We begin by describing some basic features of wavelets as a prelude to discussing the continuous wavelet transform and its properties. A wavelet is a function $\psi(x)$, centred at $x = 0$, having a few oscillations that decay to zero such that [12,13]

$$\int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x = 0. \tag{2}$$

If $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(x)$, then condition (2) is equivalent to the requirement that $\hat{\psi}(0) = 0$ [13]. While some wavelets of practical interest do not meet this requirement exactly (for example, the Morlet wavelet), $\hat{\psi}(0)$ is usually sufficiently close to zero to be of little consequence. Although there is considerable freedom in the choice of function $\psi(x)$, not every function that satisfies Eq. (2) is necessarily a good wavelet. One cycle of a sine wave, for example, makes a poor wavelet [14]. It is usual to normalise the amplitude of $\psi(x)$ so that it has unit norm; $\|\psi\| = 1$ where

$$\|\psi(x)\|^2 = \int_{-\infty}^{\infty} |\psi(x)|^2\,\mathrm{d}x. \tag{3}$$

A family of wavelets is generated from this "mother wavelet" by translations and dilations of $\psi(x)$ according to [12,13]

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right), \tag{4}$$

where $a \neq 0$ and b are real. We note that some authors choose to scale the wavelet amplitude by $1/a$ [10,11,15]; however, this choice does not preserve the energy of the wavelet with scale parameter a. Wavelets with small values of a have narrow spatial support and consequently rapid oscillations, making them well adapted to selecting high-frequency components of a signal. The converse is true for wavelets with large values of a.

The CWT of a function $f(x)$ is defined as [12,13]

$$W_f(a,b) = \int_{-\infty}^{\infty} f(x)\psi^*_{a,b}(x)\,\mathrm{d}x, \tag{5}$$

where * denotes complex conjugation. If $f(x)$ represents a row (or column) of a fringe pattern, then the CWT is a three-dimensional surface whose height (proportional to $|W_f(a,b)|$) maps the frequency content of f as a function of position b (pixel number) in position–spatial–frequency space. Since Eq. (5) is to be evaluated numerically, we note a few points of practical interest.

In the context of fringe patterns, f has a natural sampling—the pixel number—and it is sensible therefore to adopt the same sampling for the translation parameter b; $b \to n$, $n = 0, \ldots, N-1$, where N is the total number of samples. If the fringe spatial frequencies lie within a narrow range of scaling parameters $[a_{\min}, a_{\max}]$, a can be discretised by $a \to a_{\min} + k\Delta a$, $k = 0, \ldots, K$ for some suitable value of Δa. For fringes that cover a wider frequency range, a common choice is natural or log sampling for which $a = 2^m$, m an integer, so that the wavelets become [13]

$$\psi_{m,n}(x) = 2^{-m/2}\psi(2^{-m}x - n).$$

A finer sampling can be obtained by using several "voices" in each octave. This amounts to using several wavelets of the form

$$\psi^v(x) = 2^{-(v-1)/N_v}\psi(2^{-(v-1)/N_v}x), \quad v = 1, \ldots, N_v,$$

where N_v is the number of voices per octave [13,14].

A wavelet that finds frequent application is the Morlet wavelet [13]

$$\psi(x) = \pi^{-1/4}\exp(\mathrm{i}\omega_0 x)\exp(-x^2/2), \tag{6}$$

where $\omega_0 = \pi(2/\ln 2)^{1/2} \approx 5.34$. In practice, we often set $\omega_0 = 5$. This wavelet is sometimes referred to in the literature as the Gabor wavelet since it bears a resemblance to the reproducing kernel of the Gabor transform [12].

Two other wavelets of practical interest are the Mexican hat function (or Laplacian-of-Gaussian)

$$\psi(x) = \frac{2}{\sqrt{3}\pi^{1/4}}(1-x^2)\exp(-x^2/2),$$

and the Paul wavelet [16]

$$\psi(x) = \frac{2^n n!(1-\mathrm{i}x)^{-(n+1)}}{2\pi\sqrt{(2n)!/2}},$$

where n is the order of the Paul wavelet.

3. Scalograms and space–frequency resolution

The space–frequency resolution of the CWT, Eq. (5), depends on the spread of $\psi_{a,b}(x)$ in the spatial and frequency domains. If $\psi(x)$ is centred at $x = 0$, as supposed, then $\psi_{a,b}(x)$ is centred at $x = b$. The spread