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A Bayesian approach to estimation of dynamic models with small and large number of heterogeneous players and latent serially correlated states☆

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ABSTRACT

We propose a Bayesian approach to estimating dynamic models that can have state variables that are latent, serially correlated, and heterogeneous. Our approach employs sequential importance sampling and is based on deriving an unbiased estimate of the likelihood within a Metropolis chain. Under fairly weak regularity conditions unbiasedness guarantees that the stationary density of the chain is the exact posterior, not an approximation. Results are verified by Monte Carlo simulation using two examples. The first is a dynamic game of entry involving a small number of firms whose heterogeneity is based on their current costs due to feedback through capacity constraints arising from past entry. The second is an Ericson and Pakes (1995) style game with a large number of firms whose heterogeneity is based on the quality of their products with firms competing through investment in product quality that affects their market share and profitability. Our approach facilitates estimation of dynamic games with either small or large number of players whose heterogeneity is determined by latent state variables, discrete or continuous, that are subject to endogenous feedback from past actions.

1. Introduction

In many economic applications a researcher is interested in estimating a dynamic model where some of the agent specific states may be subject to feedback from past actions (e.g., Rust (1987)), partially observed and heterogeneous (e.g., Keane and Wolpin (1997)) or serially correlated (e.g., Miller (1984), Pakes (1986)).[1] In this paper, we propose a Bayesian approach to estimating dynamic models with a partially observed state that has a Markovian representation of the dynamics and an algorithm to solve the model including those with serially correlated, endogenous, heterogeneous, state variables. The method uses sequential importance sampling (particle filter) to compute an unbiased estimate of the likelihood within a Metropolis MCMC chain. Here, unbiasedness means that expectation is with respect to the uniform draws that are used to compute the particle filter with all else held fixed. Unbiasedness guarantees that the stationary density of the chain is the exact posterior, not an approximation. The regularity conditions are weak and should be satisfied by most stationary, parametric models of interest.

Our approach contributes to the literature in several ways. We derive the unbiasedness of the estimator of the likelihood that leads to an exact posterior. This allows for tractable computation and feasible estimation of a dynamic model. In addition the latent state variables can be either discrete or continuous. Moreover, it permits endogenous feedback of past actions on the latent state variables that leads to heterogeneity among the players. We illustrate our method with two examples. The first example in Section 6.1 is based on a dynamic model of entry developed in Gallant et al. (2017). This is a dynamic discrete game in which a firm's cost of production is a continuous state variable that is serially persistent, unobserved (to the researcher), and endogenous to past actions. The endogenous feedback arises because past entry in markets for other products creates a capacity constraint that affects the costs of entering a market in the current period. This

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[1] For more elaborate discussion on the motivations and rationale for estimating such dynamic models see e.g., Rust (1994), Keane (2010) and Wolpin (2013).

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endogenous evolution of costs induces heterogeneity among firms over time in the model. Our second example, in Section 6.2, is an Ericson and Pakes (1995) style model from Weintraub et al. (2010) with a large number of firms whose heterogeneity is based on the quality of their products. The firms compete by investing in product quality that in turn affects their market share, revenues and profits.

Empirical models of static and dynamic games are differentiated by their information structures. A unified approach for dynamic models of incomplete information has been developed by Hu and Shum (2012). Blevins (2016) also uses sequential importance sampling for estimating games of incomplete information, while Blevins et al. (2017) allow for two-step estimation of such games with endogenous feedback. In our paper, we focus on dynamic, complete information games because of the paucity of the literature on estimation of such games. These games typically require the use of a combinatorial algorithm to search for an equilibrium instead of the continuous fixed point mapping used in incomplete information models. Unlike games of incomplete information, the complete information assumption requires that no player has any private information. However, it allows substantial unobserved heterogeneity at the level of the firms because the researcher does not observe all the information that the players have. In contrast, games of incomplete information require there to be no difference between what is observed by the players and the researcher. While static games of complete information have been estimated by, e.g., Bresnahan and Reiss (1991), Berry (1992), Ciliberto and Tamer (2009) and Bajari et al. (2010), to our knowledge, we are the first to study dynamic games of complete information.

We prove the unbiasedness of an estimator of a likelihood obtained via particle filtering under regularity conditions that allow for endogenous feedback from the observed measurements to the dynamic state variables. Endogenous feedback is the feature that distinguishes this paper from the bulk of the particle filter literature. We establish our results by means of a recursive setup and an inductive argument that avoids the complexity of ancestor tracing during the resampling steps. This process allows elegant, compact proofs.

The remainder of this paper is organized as follows. We discuss the related literature in Section 2. Section 3 describes the games to which our results apply. An algorithm for unbiased estimation of a likelihood is proposed and unbiasedness is proved in Section 4. The MCMC estimation algorithm is presented in Section 5. Two examples are described in Section 6: the first has a small number of players, the second a large number of players. Simulation results for the two examples are presented in Section 7. Section 8 concludes.

2. Related literature

There is a growing literature on the estimation of games. Some of this literature focuses on games of incomplete information, either static (e.g., Haile et al. (2008), Ho (2009)) or dynamic (e.g., Aguirregabiria and Mira (2007), Bajari et al. (2007), Pakes et al. (2007)). The literature on estimating games of incomplete information is largely based on the two-step conditional choice probability estimator of Hotz and Miller (1993).[2] Arcidiacono and Miller (2011) have extended the literature on two step estimation of dynamic models of discrete choice to allow for unobserved heterogeneity in discrete types of latent states using the EM algorithm. Bayesian approaches that use MCMC for integrating out the unobserved state variables that are serially correlated over time have been developed by Imai et al. (2009) and Norets (2009). These papers focus on single agent dynamic discrete choice models with unobserved state variables. In contrast, we use sequential importance sampling to integrate out the unobserved state variables. Additionally, we are the first to apply this method to multi-agent dynamic games with strategic interaction, which are more computationally complex than single agent dynamic models.

The purely methodological papers most closely related to the econometric approach used here are Keane (1994), and more recently, Flury and Shephard (2010) and Pitt et al. (2012).[3] Fernandez-Villaverde and Rubio-Ramirez (2005) used sequential importance sampling to estimate dynamic stochastic general equilibrium models. Most of this literature, however, abstracts from the strategic interaction between agents. Ackerberg (2009) has developed a method for using importance sampling coupled with a change of variables technique to provide computational gains in estimating certain game theoretic and dynamic discrete choice models that admit a random coefficient representation.

3. The game

A prominent example of a model to which our results apply is a dynamic game of complete information which we describe next. The game consists of I players, $i = 1, \ldots, I$, who can choose action a_{it} at each time period t. Let $a_t = (a_{1t}, a_{2t}, \ldots, a_{It})$. In an entry game as in our first Monte Carlo example in Section 6.1, $a_{it} = 1$ if firm i enters at time t, if not, $a_{it} = 0$. Another example of a_{it} could be the level of investment which affects the quality of a firm's product as in an Ericson and Pakes (1995) style model, that is found in our second Monte Carlo example in Section 6.2. Time runs in discrete increments from $t = 0, \ldots, \infty$. The state vector is $x_t = (x_{1t}, x_{2t})$, where in turn $x_{1t} = (x_{1t,i}, x_{1t,-i})$ and $x_{2t} = (x_{2t,i}, x_{2t,-i})$. The actions of all players a_t and the state vector $x_t = (x_{1t}, x_{2t})$ is observable by all players. We (the researchers) only observe the actions of all players a_t and the state x_{2t} but not x_{1t}. The $x_{1t} = (x_{1t,i}, x_{1t,-i})$ is an agent specific latent (to the researcher) state that is allowed to be serially correlated. In our first Monte Carlo example in Section 6.1, this is considered to be the latent firm specific cost of production. In the our second Monte Carlo example in Section 6.2, this is the firm's product quality. The $x_{2t} = (x_{2t,i}, x_{2t,-i})$ is any observable market level or firm specific observable state, such as a market level demand or cost shifter or firm characteristic such as past market entry experience. The game is indexed by a parameter vector θ that is known to the players and which we seek to estimate.

To formalize the model further, we define the reduced form one-shot payoff function (e.g., Bresnahan and Reiss (1991), Berry (1992)) as,

$$\begin{aligned}\Pi_i(&a_{it}, a_{-it}, x_{1t,i}, x_{1t,-i}, x_{2t,i}, x_{2t,-i}, \theta)\\ &= R(a_{it}, a_{-it}, x^R_{1t,i}, x^R_{1t,-i}, x^R_{2t,i}, x^R_{2t,-i}, \theta^R)\\ &\quad - C(a_{it}, a_{-it}, x^C_{1t,i}, x^C_{1t,-i}, x^C_{2t,i}, x^C_{2t,-i}, \theta^C),\end{aligned} \tag{1}$$

where $R(\cdot)$ is a revenue function, and $C(\cdot)$ is a cost function. The state variables can affect either the revenue or cost functions, and are denoted as $x^R_{qt,j}$ and $x^C_{qt,j}$, respectively, as appropriate, for $q = 1, 2$ and $j = i, -i$. Moreover, the latent firm specific stochastic state, $x_{1t,i}$ can in principle affect either the revenue or cost function or both. In our first Monte Carlo example, it is modeled as firm specific cost of production. However, in the our second Monte

[2] The two step estimation strategy makes the restrictive assumption that there is no market or firm level unobserved heterogeneity other than an IID random shock across both time and players. This assumption rules out any dynamics in the latent state variables. Moreover, it precludes any private information that an agent might possess about its rivals that is unavailable to the researcher.

[3] See Doucet et al. (2001) and Liu (2008) for an overview and examples of other applications.