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# Utility representation of an incomplete and nontransitive preference relation ☆

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**Abstract**

The objective of this paper is to provide continuous utility representation theorems analogous to Debreu's classic utility representation theorem, albeit for preference relations that may fail to be complete and/or transitive. Specifically, we show that every (continuous and) reflexive binary relation on a (compact) metric space can be represented by means of the maxmin, or dually, minmax, of a (compact) set of (compact) sets of continuous utility functions. This notion of "maxmin multi-utility representation," generalizes the recently proposed notions of "multi-utility representation" for preorders and "justifiable preferences" for complete and quasitransitive relations. As such, our main representation theorems lead to some new characterizations of these special cases as well.



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## 1. Introduction

The notion of "utility representation" is one of the most fundamental constructs of economic theory. For a given binary relation $\mathbf{R}$ on a set $X$, this notion corresponds to finding a real function $u$ on $X$ such that

$$x \mathbf{R} y \quad \text{iff} \quad u(x) \geq u(y) \tag{1}$$

for every $x$ and $y$ in $X$. We often think of $\mathbf{R}$ as a preference relation of an individual, and interpret $u$ as a utility function that keeps record of how this agent ranks any two alternatives. Obviously, not every binary relation can be represented in this manner. In particular, such a representation is possible only if $\mathbf{R}$ is complete and transitive. The converse is not true in general, but there are a plethora of results that provide various sufficient conditions for such $\mathbf{R}$ to admit a utility representation, and when $X$ is a topological space, to guarantee that the map $u$ to be found is continuous. The most famous of these results is *Debreu's Utility Representation Theorem* which says that there is a continuous real map $u$ on $X$ such that (1) holds for each $x, y \in X$, provided that $X$ is a suitably well-behaved space (such as a separable metric space) and $\mathbf{R}$ is complete, transitive and continuous.

However, in many contexts, one needs to consider as a primitive "preference relation" a binary relation that may be neither complete nor transitive. In particular, since the seminal contributions of Aumann (1962) and Bewley (1986), many authors have argued that completeness is not a basic trait of rationality. There is now a fairly sizable literature on rational decision making with incomplete preferences in a variety of contexts, ranging from consumption choice to decision making under risk and uncertainty. And there is even a larger literature that works with nontransitive preferences. This literature has mostly a "boundedly rational" flavor, and it studies topics such as nontransitive indifferences that arise from perception difficulties (cf. Luce, 1956), or procedural decision making by using similarity comparisons or regret considerations (cf. Rubinstein, 1988 and Loomes and Sugden, 1982), or time inconsistency that arises due to relative time discounting (cf. Roelofsma and Read, 2000, and Ok and Masatlioglu, 2007), among others.[1] Besides, when we consider $\mathbf{R}$ as the preference relation of a group of individuals (as in social choice theory), it becomes only natural to allow for its lack of transitivity. We may further add to this summary by noting that in *revealed preference* theory one arrives at a "preference relation" endogenously, and in many cases of interest it is impossible to guarantee either the completeness or the transitivity of this relation. The recent literature on boundedly rational choice theory provides numerous illustrations of this situation.[2]

These considerations motivate extending Debreu's Theorem in some way to the context of binary relations that are neither complete nor transitive. Obviously, this requires us to modify the "utility representation" notion used in that theorem appropriately. In the case of incomplete, but transitive, binary relations, one such notion was introduced in Ok (2002), and then later developed in Evren and Ok (2011) and Bosi and Herden (2012). This notion is called "multi-utility representation," and corresponds to finding a *collection* $\mathcal{U}$ of real functions on $X$ such that

[1] The literatures on incomplete and nontransitive preferences is simply too large to cite here comprehensively. For numerous illustrations and citations from the first of these, we refer the reader to Evren and Ok (2011), and for those from the second to Nishimura (2015).

[2] In Eliaz and Ok (2006), for instance, revealed preference relations are incomplete, and in Cherepanov et al. (2013), they are nontransitive. On the other hand, Manzini and Mariotti (2007), Masatlioglu and Ok (2014), and Ok et al. (2015) use the revealed preference method to obtain what they call "psychological constraint relations" which need not be either complete or transitive.