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A regularized finite-element digital image correlation for irregular displacement field

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ABSTRACT

A nonlinear Tikhonov regularization scheme is developed to tackle the ill-posed finite-element digital image correlation, which aims to measure the displacement field from consequent digital images before and after deformation. The goal of this algorithm is to resolve the displacement field with fine and irregular structure without deteriorated by the measuring errors due to its ill-posedness. A Newton-type method is employed to linearize the nonlinear problem iteratively, then the Tikhonov regularization is applied to the linearized problem, with the regularization parameter adaptively chosen by the L-curve method. The proposed algorithm is verified by computer simulated input images with *a priori* displacement field. The result shows that it is capable of resolving displacement field with very fine structure in a reasonable accuracy.

1. Introduction

Digital image correlation (DIC) [1,2] is a technique which could measure the displacement field from consequent digital images before and after deformation (referred as the reference and deformed image, respectively) . It is a versatile method that works for any measuring techniques that outputs digital images, with measuring scale ranging from nanoscopic to macroscopic such as CCD camera, scanning electron microscope (SEM), atomic force microscope (AFM), scanning tunneling microscope (STM), X-ray Computerized Tomography (XCT). Due to its simple experimental setup, non-contacting measurement and high accuracy, DIC has been extensively investigated and has become a popular and powerful method for measuring surface or 3D deformation which is one of central topics in structural mechanics. Since the displacement field can be related to many mechanical parameters, DIC can be applied to evaluate some mechanical properties directly, such as the stress intensity factor [3,4].

The main idea of DIC is to select an interested subset (usually rectangle) in the reference image, then use it as a template to find its new location in the deformed image. The criterion is that the *correlation* between matching subsets in reference and deformed image should reach its maximum to make a best match. The correlation characterizes the similarity of these two subsets, it can be chosen as the cross-correlation or the sum-squared difference (and also some other variations of them [2]), depend on the input images. Then this becomes a parametric optimization problem [5], with the *correlation* as the only objective function. It could be solved efficiently by some iterative methods such as the Newton–Raphson method or the Levenberg–Marquart algorithm. For full-scope displacement field, just repeat this procedure independently for every interested subset (usually non-overlapping) in the reference image. The problem of this traditional method is that the displacement field is generally not continuous at the boundaries of subsets. Another shortcoming is that the subset size should be large (typically tens or hundreds of pixels [6,7]) to obtain robust results, which will certainly limit the resolution of the result displacement field.

To solve the problem of discontinuity of displacement field between adjacent subsets, a finite-element variation of DIC [8,9] was developed so that it can intrinsically output continuous displacement field, aiming to bridge the gap in structural mechanics between experiments and simulations. Instead of local pattern-matching individually as in traditional DIC, it employs a finite-element *mesh* strategy that considers all subsets in a global scope. The displacement field is expressed as the weighted sum of a set of shape functions corresponding to the nodes. The boundaries of elements share the same values, and the continuity can be determined by the type of finite element shape functions. Furthermore, this strategy is compatible with usual structural mechanical simulation, which makes it possible to directly compare the displacement measurement and the finite element simulation result. Due to the global consideration, it allows smaller subset size (usually down to several pixels) than the traditional DIC (typically several tens of pixels) [8]. Another virtue of this finite element approach is that it can incorporate with the enrichment strategy of extended finite element method [10], which allows more efficient and accurate representation of interfaces such as cracks, holes and inclusions.

This finite element variation of DIC solves the discontinuity problem of the traditional DIC, but it does not solve the problem of

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the subset size completely. The result displacement field in the element is usually constant or bi-linear depending on the order of finite element basis, hence it cannot resolve fine displacement pattern smaller than the element size. In some practical cases, the displacement field may have very fine and irregular structure, especially for heterogeneous and porous materials, such as concrete, foam [11,12] and wood [13]. The irregular displacement field is very important for identifying the contribution of different components, and will also be very helpful for the research of their constitutive relation. Thus, for these materials, finer elements should be used. However, due to the ill-posedness of the problem, the *naive* solution of maximizing the correlation will become less reliable, and the accuracy will also decrease. When the element size is down to a few pixels, the error becomes dominating and the result will be totally meaningless. Therefore the regularization is necessary to stabilize the solution. For traditional DIC, Cofaru et al. [14,15] improved the Newton–Raphson method by the adaptive spatial regularization which allows more robust strain calculation. For the finite-element DIC, a regularized digital volume correlation based on the equilibrium gap method was developed by Leclerc [16] to output displacement field in voxel scale. However, this method may require some *a priori* information to take proper regularization, which is usually not available for displacement field with irregular structure.

In this paper, an iterative Tikhonov regularization will be applied to the nonlinear finite-element DIC problem. The regularization parameter is chosen adaptively by L-curve method without *a priori* information. The proposed method will be tested by computer simulated input images with various kinds of displacement field discretized by different element sizes. The results with and without regularization will be computed, which will give a quantitative comparison of the regularization effect.

2. Basic principles of finite-element DIC

The start point of finite-element DIC is the conservation of optical flow, i.e., two points in the image before and after deformation (i.e., the reference image $f(\mathbf{p})$ and deformed image $g(\mathbf{p})$, respectively) corresponding to the same physical entity have the same luminance:

$$g(\mathbf{p}) = f(\mathbf{p}+\mathbf{u}(\mathbf{p})) \tag{1}$$

The task of finite-element DIC is to measure the displacement field $\mathbf{u}(\mathbf{p})$ from the knowledge of f and g as accurately as possible. Since this is a set of nonlinear equations, it can be iteratively solved by nonlinear Newton-type method. The main idea is to linearize the equation in the neighbor of current approximation $\mathbf{u}_0$ (and current error $\delta\mathbf{u} = \mathbf{u}-\mathbf{u}_0$) using Taylor expansion of first order (providing f is differentiable):

$$\begin{aligned} f(\mathbf{p}+\mathbf{u}) &= f(\mathbf{p}+\mathbf{u}_0+\delta\mathbf{u}) \\ &\approx f(\mathbf{p}+\mathbf{u}_0)+\nabla f(\mathbf{p}+\mathbf{u}_0)\cdot\delta\mathbf{u} \end{aligned} \tag{2}$$

If $\delta\mathbf{u}$ can be discretized by finite element basis:

$$\delta\mathbf{u}(\mathbf{p}) = \sum_{i,\alpha}\delta\mathbf{x}_{i\alpha}\phi_{i\alpha}(\mathbf{p}) \tag{3}$$

where the subscripts i, α denote the α component of displacement field of node i. Then the problem can be written as the minimization of residual integration over interested region:

$$\eta^2 = \iint_\Omega \left(\nabla f(\mathbf{p}+\mathbf{u}_0)\cdot\sum_{i,\alpha}\delta\mathbf{x}_{i\alpha}\phi_{i\alpha}(\mathbf{p})+f(\mathbf{p}+\mathbf{u}_0)-g(\mathbf{p})\right)^2 \mathrm{d}\mathbf{p} \tag{4}$$

the solution can be carried out by a simple linear system:

$$\mathbf{A}\,\delta\mathbf{x} = \delta\mathbf{b} \tag{5}$$

in which

$$\mathbf{A}_{i\alpha j\beta} = \iint_\Omega \phi_{i\alpha}\nabla_\alpha f(\mathbf{p}+\mathbf{u}_0)\phi_{j\beta}\nabla_\beta f(\mathbf{p}+\mathbf{u}_0)\,\mathrm{d}\mathbf{p} \tag{6}$$

$$\delta\mathbf{b}_{i\alpha} = \iint_\Omega (g(\mathbf{p})-f(\mathbf{p}+\mathbf{u}_0))\phi_{i\alpha}\nabla_\alpha f(\mathbf{p}+\mathbf{u}_0)\,\mathrm{d}\mathbf{p} \tag{7}$$

As soon as $\delta\mathbf{u}$ is obtained, $\mathbf{u}_0+\delta\mathbf{u}$ should yield a better approximation of $\mathbf{u}$. The procedure will be repeated with this new approximation until convergence is obtained.

The algorithm proposed here can be considered as the finite-element version of iterative least square approach developed by Pan et al. [17]. When the convergence can be guaranteed, it usually can give better results than just solving (5) once, which is actually the first order approximation.

However, the linear system (5) may be quite ill-posed, which means the solution is very sensitive to the measuring errors in $\delta\mathbf{b}$, especially when the element size is small, thus solving it directly may result in quite inaccurate solution. Therefore, the regularization technique is necessary for a robust estimation of the displacement field.

3. Tikhonov regularization of finite-element DIC

3.1. Iterative regularization of nonlinear equation

The ill-posed nonlinear operator equation

$$\mathbf{F}(\mathbf{x}) = \mathbf{y} \tag{8}$$

emerges in many applications [18]. Its ill-posedness means that the solution $\mathbf{x}$ does not depend on $\mathbf{y}$ continuously, or even not uniquely. Thus the *naive* solution by directly solving (8) would be quite sensitive to $\mathbf{y}$. However, in most cases, $\mathbf{y}$ is more or less contaminated by measuring errors, which would inevitably affect the solution. When the problem is severely ill-posed, even tiny errors would deteriorate the solution drastically, making it totally meaningless. Therefore, special techniques, so-called regularization method, should be employed to obtain stable approximation of $\mathbf{x}$.

A common regularization technique for nonlinear equations is based on the Newton-type iteration. The key idea of Newton-type method is to iteratively linearize the nonlinear equation $F(\mathbf{x}) = \mathbf{y}$ around current approximate value $\mathbf{x}_k^\delta$, then solving the linearized equations

$$F'(\mathbf{x}_k^\delta)(\mathbf{x}_{k+1}^\delta-\mathbf{x}_k^\delta) = \mathbf{y}-F(\mathbf{x}_k^\delta) \tag{9}$$

will yield next approximate value $\mathbf{x}_{k+1}^\delta$. When the convergence is guaranteed, $\mathbf{x}_{k+1}^\delta$ will be closer to the true value $\mathbf{x}^*$ than $\mathbf{x}_k^\delta$: $\|\mathbf{x}_{k+1}^\delta-\mathbf{x}^*\| \le \|\mathbf{x}_k^\delta-\mathbf{x}^*\|$. After many iterations the difference should be small enough and finally $\mathbf{x}_k$ gives an accurate estimation of the true value $\mathbf{x}^*$. For finite-element DIC, the linearized equation is just $\mathbf{A}\delta\mathbf{x} = \delta\mathbf{b}$. For practical implementation, since the regularization operator is with respect to $\mathbf{u}$ instead of $\delta\mathbf{u}$, the equation $\mathbf{A}\delta\mathbf{x} = \delta\mathbf{b}$ can be transformed to another more convenient form:

$$\mathbf{A}\mathbf{x} = \mathbf{A}(\mathbf{x}_0+\delta\mathbf{x}) = \mathbf{A}\mathbf{x}_0+\delta\mathbf{b} = \mathbf{b} \tag{10}$$

in which $\mathbf{x}_0$, $\mathbf{x}$ are the node weights of $\mathbf{u}_0$, $\mathbf{u}$ as in (3).

The Tikhonov regularization is probably one of the most popular regularization method for solving both linear and nonlinear ill-posed problems [19,20]. Instead of solving (10) directly, Tikhonov regularization stabilizes the solution by minimize the functional with an additional penalty term:

$$\|\mathbf{b}-\mathbf{A}\mathbf{x}\|^2+\lambda\|\mathbf{L}(\mathbf{x})\|^2, \quad \lambda > 0 \tag{11}$$

where $\mathbf{L}$ denotes the regularization operator, generally characterizes the smoothness of the solution. The penalty weight λ, often referred