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Local smoothness and the price of anarchy in splittable congestion games ☆

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Abstract

Congestion games are multi-player games in which players' costs are additive over a set of resources that have anonymous cost functions, with pure strategies corresponding to certain subsets of resources. In a *splittable* congestion game, each player can choose a convex combination of subsets of resources. We characterize the worst-case price of anarchy — a quantitative measure of the inefficiency of equilibria — in splittable congestion games. Our approximation guarantee is parameterized by the set of allowable resource cost functions, and degrades with the "degree of nonlinearity" of these cost functions. We prove that our guarantee is the best possible for every set of cost functions that satisfies mild technical conditions. We prove our guarantee using a novel "local smoothness" proof framework, and as a consequence the guarantee applies not only to the Nash equilibria of splittable congestion games, but also to all correlated equilibria.

JEL classification: C72; C73; D62; R41

Keywords: Atomic; Congestion; Correlated equilibrium; Price of anarchy; Splittable

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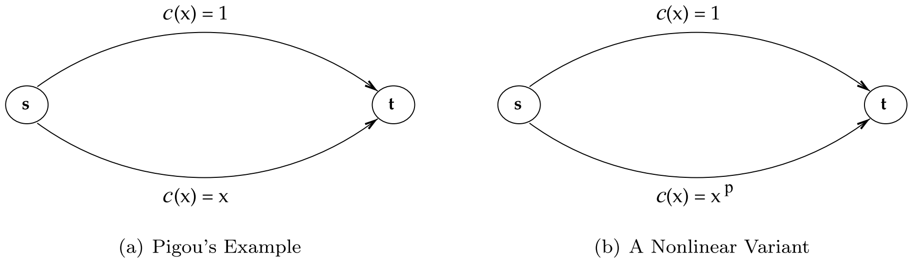

(a) Pigou's Example (b) A Nonlinear Variant

Fig. 1. The price of anarchy grows with the "degree of nonlinearity" of the resource cost functions.

1. Introduction

Congestion games play a central role in the theory of worst-case approximation guarantees for game-theoretic equilibria. They are expressive enough to capture a number of otherwise unrelated applications — including routing, network design, oligopoly models, and the migration of species [2,18,19,24,28] — yet structured enough to permit interesting theoretical guarantees. In the standard model introduced by Rosenthal [24], there is a ground set of resources, and each player selects a subset of them (e.g., a path in a network). Each resource has a univariate cost function that depends on the load induced by the players that use it, and each player strives to minimize the sum of the resources' costs in its chosen strategy (given the strategies chosen by the other players). Because of congestion externalities — that is, because each player ignores the extra cost its action imposes on the other players — Nash equilibria of congestion games typically do not minimize the joint cost of the players.

We study the *splittable* variant of congestion games, where each player has a *weight* w_i and a list of available strategies (each a subset of resources), and each player chooses how to split fractionally its weight over its strategies.[3] The splittable model is more appropriate than the traditional "unsplittable" model in some applications, such as multipath routing in networks. Indeed, in the computer networking literature, the splittable model was studied a decade prior to the unsplittable model, beginning with [22]. The splittable model also arises naturally when studying coalitions of players in nonatomic congestion games, where there is a continuum of players [7,8,14,16].

The goal of this paper is to quantify the inefficiency of Nash equilibria in splittable congestion games. To measure inefficiency, we use the *price of anarchy (POA)* [17]: the worst-case ratio between the sum of players' costs in a Nash equilibrium and in a minimum-cost outcome. To develop intuition for the POA in congestion games, we informally review a simple example, essentially due to Pigou [23]. Consider the two-vertex, two-edge network shown in Fig. 1(a). Resources correspond to edges, and strategies correspond to s–t paths. Assume that there is a very large number of players, each with negligible weight, with the total weight of all players summing to 1. Each edge is labeled with a cost function, describing the cost incurred by traffic on that edge, as a function of the sum of the weights of the players on that edge. With negligible-size players, the lower edge is a dominant strategy for every player. Thus, there is a Nash equilibrium in which the average player cost is 1. On the other hand, in an outcome where the players are

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[3] Deterministically spreading weight over multiple strategies is *not* equivalent to probabilistically selecting a single strategy, except in the trivial case of load-independent resource cost functions.

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