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Unit root vector autoregression with volatility induced stationarity

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ABSTRACT

We propose a discrete-time multivariate model where lagged levels of the process enter both the conditional mean and the conditional variance. This way we allow for the empirically observed persistence in time series such as interest rates, often implying unit-roots, while at the same time maintain stationarity despite such unit-roots. Specifically, the model bridges vector autoregressions and multivariate ARCH models in which residuals are replaced by levels lagged. An empirical illustration using recent US term structure data is given in which the individual interest rates are found to have unit roots, have no finite first-order moments, but remain strictly stationary and ergodic. Moreover, they co-move in the sense that their spread has no unit root. The model thus allows for volatility induced stationarity, and the paper shows conditions under which the multivariate process is strictly stationary and geometrically ergodic. Interestingly, these conditions include the case of unit roots and a reduced rank structure in the conditional mean, known from linear co-integration. Asymptotic theory of the maximum likelihood estimators for a particular structured case (so-called self-exciting) is provided, and it is shown that $\sqrt{T}$-convergence to Gaussian distributions apply despite unit roots as well as absence of finite first and higher order moments. Monte Carlo simulations illustrate the asymptotic theory.

1. Introduction and summary

This paper presents a new multivariate time series model which captures important stylized facts of the dynamics of term structure data. In particular the model allows for the typically observed persistence in interest rates often detected as unit-roots in the conditional mean in empirical analyses. But contrary to classic autoregressive models, the variables here enter the conditional variance as well. This can induce stationarity, such that despite unit-roots the multivariate process is stationary. In short, the model allows for co-movement of individual series such that for example spreads have no unit-roots, while at the same time the individual time series are allowed to have unit-roots and to be persistent but remain stationary. We present theory for inference as well as discuss properties of the applied model, and moreover demonstrate by our empirical analysis that the model captures surprisingly well dynamic features of US term structure data. Regarding the results on inference we show that standard asymptotic inference applies despite the fact that an immediate implication of the unit-roots present in the model is that the processes will have fat tails and only finite low order moments.

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Our insistence on allowing for unit-roots is based on the rich literature in econometrics from which it is by now a stylized empirical fact that term structure data, as well as other financial economic time series, are persistent and appear to have unit-roots when modeled as autoregressive (AR) processes. However, the implied non-stationarity is often questioned from an economic, or finance, point of view, and alternative models which allow unit-root in the conditional mean, but are stationary due to the formulation of the conditional volatility have been proposed in the literature. A large part of such literature deals with continuous time models, where a key example is the celebrated Cox–Ingersoll–Ross model, cf. Cox et al. (1985), where both drift and volatility terms are functions of the level of the continuous time process. Our proposed model also fits well within the rich term structure literature, both in continuous as well as in discrete time, see for example Gourieroux et al. (2002), Le et al. (2010) and Carta et al. (2008) for affine term structure models.

With respect to discrete time series models, the proposed model bridges two different time series modeling approaches: *co-integration analysis*, which is multivariate and allows for unit-roots or reduced rank, and the more recent *double autoregressive modeling*, which so far has been univariate. In the co-integrated vector AR models, see Johansen (1996), unit-roots, or equivalently, reduced rank in the conditional mean imply that the stochastic trends are non-stationary and have random walk behavior. This contrasts the univariate double autoregressive (DAR) models where a unit-root does not necessarily imply non-stationarity as lagged values of the process enter the conditional variance, see e.g. Borkovec and Klüppelberg (2001) and Ling (2004). To fix ideas consider initially the univariate DAR of order one as given by,

$$x_t = x_{t-1} + \left[\sqrt{\omega_{xx} + \phi_{xx}^2 x_{t-1}^2}\right] z_{xt}, \tag{1}$$

where z_{xt} is i.i.d. $N(0, 1)$, and $\omega_{xx} > 0$, $\phi_{xx}^2 > 0$. Despite the unit-root in the conditional mean, the process is strictly stationary provided $0 < \phi_{xx}^2 \lesssim 2.42$, which contrasts the case where squared lagged differences enter the conditional variance in which case unit-roots indeed imply non-stationarity, see Lange et al. (2011). In terms of our proposed term structure modeling the univariate unit-root DAR can be thought of as a model for the short term interest rate, driving the level and volatility of interest rates with different maturities. Consider thus next y_t defined as the spread between an interest rate with long maturity and the short term rate x_t, with the dynamics of y_t given by,

$$y_t = \rho y_{t-1} + \left[\sqrt{\omega_{yy} + \phi_{yy}^2 y_{t-1}^2 + \phi_{yx}^2 x_{t-1}^2}\right] z_{yt}, \tag{2}$$

with z_{yt} i.i.d. $N(0, 1)$, and ω_{yy}, ϕ_{yy}^2, $\phi_{yx}^2 > 0$, while the autoregressive parameter $\rho \in \mathbb{R}$. This way, the lagged short term rate x_t, enters as a variance factor of the spread y_t. That is, a stationary factor with a unit-root enters the conditional variance of the spread, where if $|\rho| < 1$ the interest rates are co-moving. Note that, as will be detailed in Section 3 for general versions of the process which include the above, that the process nests random-walk type behavior (as ϕ_{xx} approaches zero), and, despite the Gaussian innovations ($z_{it}, i = y, x$), the joint process will have Paretian, or fat, tail like behavior. That is, the model indeed matches characteristics of financial data. Moreover, note that in terms of the affine term structure framework of Le et al. (2010) also the joint process $(y_t, x_t)'$ can be used as a multivariate (unobservable) factor.

Within the unit-root and co-integration literature much attention has been devoted to the inclusion and role of constant terms. A key problem in this strand of literature is that a constant term μ may induce a linear trend due to the implied aggregation as caused by the unit-roots. This has lead to various ways of including restricted constants, and other deterministic terms, in multivariate co-integrated vector AR models, cf. Johansen (1996). However, as we show below, we do avoid such issues here and can include an unrestricted constant vector μ in our multivariate model. This is novel in the framework of nonlinear time series. In terms of Eq. (1), we show that adding a constant term, μ_x say, on the right hand side does not imply a linear trend, nor that the properties of x_t in terms of stationarity and (geometric) ergodicity are changed, despite the unit-root in x_t.

For an introduction to the recent univariate DAR models, inference and estimation have been explored for univariate DAR models in Ling (2004, 2007) and Ling and Li (2008), while extremal and tail behavior have been analyzed for DAR models of order one in Borkovec (2000), Borkovec and Klüppelberg (2001) and Klüppelberg and Pergamenchtchikov (2004). As also used in the mentioned references DAR processes with Gaussian innovations may be restated as random coefficient autoregressions (RCAR), for which Aue et al. (2006) and Berkes et al. (2009) provide results, as well as key references, on estimation and inference. In fact, the RCAR approach is applied in Fong and Li (2004), where co-integration is discussed with random coefficients. The parametrization in Fong and Li (2004) of the conditional variance is quite different when compared to ours, and moreover Fong and Li (2004) apply a local approach where the conditional variance parameters loading the levels vanish at the rate of T, where T denotes the number of observations. Klüppelberg and Pergamenchtchikov (2007) study extremal behavior of a class of multivariate RCAR processes with finite second order moments, which excludes the unit-roots which is a main interest here.

The paper is structured as follows: In the next section we describe in detail our proposed multivariate model. The model is a vector AR model with reduced rank structure in the conditional mean, allowing for unit-roots, while the conditional variance is cast in line with multivariate BEKK ARCH models but in levels. Next, we derive conditions for stationarity, geometric ergodicity and existence of moments, where it is emphasized that the unit-roots imply that the process in general, while being stationary, will only have finite small order moments. Asymptotic theory of the maximum likelihood estimators for the applied US term structure model is given. It is found that despite the fact that the processes lack finite even second and first order moments,

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