Journal of Mathematical Economics 76 (2018) 21–32

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

On the (non-)differentiability of the optimal value function when the optimal solution is unique ☆

Daisuke Oyama [a,*], Tomoyuki Takenawa [b]

[a] Faculty of Economics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
[b] Faculty of Marine Technology, Tokyo University of Marine Science and Technology, 2-1-6 Etchu-jima, Koto-ku, Tokyo 135-8533, Japan

ARTICLE INFO

Article history:
Received 30 January 2017
Received in revised form 31 January 2018
Accepted 15 February 2018
Available online 26 February 2018

Keywords:
Parameterized optimization
Sensitivity analysis
Value function
Envelope theorem

ABSTRACT

We present examples of a parameterized optimization problem, with a continuous objective function differentiable with respect to the parameter, that admits a unique optimal solution, but whose optimal value function is not differentiable. We also show independence of Danskin's and Milgrom and Segal's envelope theorems.

1. Introduction

Parameterized optimization problems are ubiquitous in economics, from classical price theory to dynamic macroeconomics, game theory, mechanism design, and so on. There, the envelope theorem serves as a standard tool in understanding the marginal effects of changes in the parameter, such as price or technology, on the value of the optimal choice of the agents in the model. While textbook envelope theorems usually only derive a formula ("envelope formula") that the derivative of the value function, the optimal value as a function of the parameter, should satisfy under the (often implicit) assumption that the value function is differentiable, a rigorous statement of the theorem also describes a sufficient condition on the primitives under which this assumption holds true. The latter issue, the differentiability of the value function, is what we are concerned with in this paper.

We consider the following setting. Let X be a nonempty topological space (the choice set), and $A \subset \mathbb{R}$ a nonempty open set (the parameter space). The objective function $f : X \times A \to \mathbb{R}$ is to be maximized with respect to $x \in X$, given $\alpha \in A$. The optimal value function is given by

$$v(\alpha) = \sup_{x \in X} f(x, \alpha),$$

associated with the optimal solution correspondence

$$X^*(\alpha) = \{x \in X \mid f(x, \alpha) = v(\alpha)\},$$

where we assume that the partial derivative of f with respect to α, f_α, exists and that $X^*(\alpha) \neq \emptyset$ for all $\alpha \in A$. We are interested in the differentiability of the value function v (in the classical sense, rather than notions such as directional differentiability or subdifferentiability).

1. *If v is assumed to be differentiable* at $\alpha = \bar{\alpha}$, then it is easy to derive the envelope formula: for any $\bar{x} \in X^*(\bar{\alpha})$,

   $$v'(\bar{\alpha}) = f_\alpha(\bar{x}, \bar{\alpha}).$$

   Indeed, fix any $\bar{x} \in X^*(\bar{\alpha})$. Then the function $g(\alpha) = f(\bar{x}, \alpha) - v(\alpha)$, which is differentiable at $\bar{\alpha}$, is maximized at $\bar{\alpha}$, so the first-order condition $g'(\bar{\alpha}) = 0$ yields the formula.
2. One can easily construct an example in which the value function is not differentiable *when there are more than one solutions*. For example, let $X = \mathbb{R}$ and $A = (-1, 1)$, and consider

   $$f(x, \alpha) = -\frac{1}{4}x^4 - \frac{\alpha}{3}x^3 + \frac{1}{2}x^2 + \alpha x - \frac{1}{4},$$

   with $f_x(x, \alpha) = -(x + 1)(x + \alpha)(x - 1)$. Then we have

   $$v(\alpha) = \frac{2}{3}|\alpha|, \quad X^*(\alpha) = \begin{cases} \{-1\} & \text{if } \alpha < 0, \\ \{-1, 1\} & \text{if } \alpha = 0, \\ \{1\} & \text{if } \alpha > 0, \end{cases}$$

   where v is not differentiable at $\alpha = 0$, for which there are two optimal solutions.

☆ We would like to thank Atsushi Kajii, Kazuya Kamiya, Jan Werner, and anonymous referees for helpful comments and discussions.

* Corresponding author.
E-mail addresses: oyama@e.u-tokyo.ac.jp (D. Oyama), takenawa@kaiyodai.ac.jp (T. Takenawa).
URLs: http://www.oyama.e.u-tokyo.ac.jp (D. Oyama), http://www2.kaiyodai.ac.jp/~takenawa (T. Takenawa).

https://doi.org/10.1016/j.jmateco.2018.02.004
0304-4068/

3. The question we ask in this paper is: is the value function always differentiable *when the optimal solution is unique* (and the objective function is continuous and the solution correspondence admits a continuous selection)? The answer is no: we present in Section 2 an example in which f_α exists and X^* is a single-valued continuous function, but nevertheless v is not differentiable at some $\bar{\alpha}$ (Example 2.1). The main feature in our example is that f_α is not continuous at $(x, \alpha) = (\bar{x}, \bar{\alpha})$, where $\{\bar{x}\} = X^*(\bar{\alpha})$. In fact, if X^* admits a selection continuous at $\bar{\alpha}$ and f_α is continuous jointly in (x, α) at $(\bar{x}, \bar{\alpha})$, then v must be differentiable at $\bar{\alpha}$ (Proposition 2.1).

Numerous forms of sufficient conditions for the differentiability of the value function have been obtained in the literature. In Section 3, we discuss the results by Danskin (1966, 1967) and Milgrom and Segal (2002). Danskin's theorem also assumes the continuity of f_α, and when applied to the case where the optimal solution is unique, his assumptions are slightly stronger than those in Proposition 2.1 mentioned above, while they are not nested in general. Our Example 2.1 illustrates that the continuity of f_α is indispensable also in Danskin's theorem.

Milgrom and Segal (2002, Theorem 3) provide a sufficient condition in terms of the equidifferentiability of the objective function f. It turns out that our example does not satisfy this condition. We present examples that illustrate that neither of the continuity of f_α at $(\bar{x}, \bar{\alpha})$ and the equidifferentiability of $\{f(x, \cdot)\}_{x\in X}$ implies the other, showing that the conditions in Danskin's theorem, or our Proposition 2.1, and those in Milgrom and Segal's theorem are independent from each other.

It has been known that certain convexity/concavity conditions allow the differentiability of the value function. For instance, the support function of a closed convex set in a finite-dimensional space, examples of which include the profit, cost, and expenditure functions in price theory, is differentiable if and only if the maximum (or minimum) is attained at a single point; see Mas-Colell et al. (1995, Proposition 3.F.1) or Rockafellar (1970, Corollary 25.1.3). In this case, the partial derivative of the objective function with respect to the parameter is clearly continuous. In fact, we show in Section 5.1 that this theorem, well known from convex analysis, also follows from (a multidimensional parameter version of) our envelope theorem Proposition 2.1, despite the possible unboundedness of the choice set. If the objective function f is concave jointly in the choice variable x and the parameter α and if f_α exists, then the value function, which is necessarily concave, is always differentiable; see Hogan (1973), Benveniste and Scheinkman (1979), or Milgrom and Segal (2002, Corollary 3). For this result, topology is not needed for the choice set (only being a convex subset of a linear space), and hence the continuity of f_α is irrelevant.

In Section 4, we extend our example to optimization problems with inequality constraints that vary with the parameter α. We provide examples with a binding constraint in which the Lagrange function L is differentiable in α and the optimal solution and the Kuhn–Tucker vector (which constitute a saddle point of L) are unique and continuous in α, but the optimal value function is not differentiable. Again, in these examples, L_α fails to be continuous: in fact, if a function $L(x, y, \alpha)$ has a saddle point selection $(\bar{x}(\alpha), \bar{y}(\alpha))$ that is continuous in α and L_α is continuous in (x, y, α), then its saddle value function $L(\bar{x}(\alpha), \bar{y}(\alpha), \alpha)$ is differentiable in α (Proposition 4.1). We also observe that if the value function of the constrained problem is concave, then the existence of a continuous selection of Kuhn–Tucker vectors is sufficient for the differentiability (Proposition 4.3).[1]

In Section 5, we apply our analysis within the context of price theory. Section 5.1 considers the profit function (or the support function) for a closed convex production set, where, as mentioned, we present a proof for its differentiability that uses our Proposition 2.1. Section 5.2 concerns the value functions for consumption choice. We first state envelope theorems derived from Proposition 2.1 for the indirect utility function and the expenditure function: if the utility function is continuous and locally nonsatiated and has partial derivatives which are continuous and nonvanishing at the optimum, then these functions are differentiable whenever the optimal solutions are unique. Then, extending our main example into this framework, we construct an example of a continuous utility function with positive partial derivatives for which, for some fixed price vector, the Walrasian and Hicksian demands are unique and continuous in wealth w and required utility u, but the indirect utility and expenditure functions fail to be differentiable in w and u, respectively. In this example, the partial derivatives of the utility function are not continuous, demonstrating that the continuous differentiability condition cannot be replaced with the mere existence of partial derivatives for the differentiability of the value functions in this case as well.

[1] This result extends Corollary 3 in Milgrom and Segal (2002) to the case of parametric constraints. See also Marimon and Werner (2016).

2. (Non-)Differentiability of the value function

Let X be a nonempty topological space, and $A \subset \mathbb{R}$ a nonempty open set. Given the objective function $f : X \times A \to \mathbb{R}$, we consider the optimal value function

$$v(\alpha) = \sup_{x\in X} f(x, \alpha),$$

associated with the optimal solution correspondence

$$X^*(\alpha) = \{x \in X \mid f(x, \alpha) = v(\alpha)\}.$$

We are interested in the differentiability of v when X^* is point-valued. We first state a sufficient condition of direct relevance for our study.

Proposition 2.1. *Assume that*

(a) *X^* has a selection x^* continuous at $\bar{\alpha}$, and*
(b) *f is differentiable in α in a neighborhood of $(x^*(\bar{\alpha}), \bar{\alpha})$, and f_α is continuous in (x, α) at $(x^*(\bar{\alpha}), \bar{\alpha})$.*

Then v is differentiable at $\bar{\alpha}$ with $v'(\bar{\alpha}) = f_\alpha(x^*(\bar{\alpha}), \bar{\alpha})$.

Assumption (a) holds if X^* is nonempty-valued and upper semicontinuous (which holds true, e.g., when f is continuous and X is compact) and $X^*(\bar{\alpha})$ is a singleton, in which case any selection of X^* is continuous at $\bar{\alpha}$.

A version of this proposition is found in the lecture notes by Border (2015, Corollary 299). For completeness, we present the proof in Appendix A.1.

Our main observation in this paper is that, even when there is a unique optimal solution, the differentiability of v may fail if one drops the continuity of f_α.

Proposition 2.2. *There exists a continuous function $f : X \times A \to \mathbb{R}$ such that*

(a) *$X^*(\alpha)$ is a singleton for all α and is continuous in α (as a single-valued function), and*
(b) *f is differentiable in α,*

but v is not differentiable at some α.

In the following, we present an example of such a function f.