Journal of Mathematical Economics 75 (2018) 108–115

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Essential equilibrium in normal-form games with perturbed actions and payoffs[☆]

Oriol Carbonell-Nicolau [*], Nathan Wohl

Department of Economics, Rutgers University, 75 Hamilton Street, New Brunswick, NJ 08901, USA

ARTICLE INFO

Article history:
Received 22 February 2017
Received in revised form 10 January 2018
Accepted 22 January 2018
Available online 2 February 2018

Keywords:
Discontinuous normal-form game
Equilibrium refinement
Essential equilibrium
Equilibrium existence
Generalized payoff security

ABSTRACT

A Nash equilibrium of a normal-form game G is essential if it is robust to perturbations of G. A game is essential if all of its Nash equilibria are essential. This paper provides conditions on the primitives of a (possibly) discontinuous game that guarantee the generic existence of essential games. Unlike the extant literature, the present analysis allows for perturbations of the players' action spaces, in addition to the standard payoff perturbations.

1. Introduction

A Nash equilibrium of a normal-form game G is essential if it is robust to perturbations of G. For generic games in the collection of all finite-action games with fixed action spaces, all Nash equilibria are essential (*cf.* Wu and Jiang, 1962). This result has been extended to infinite-action games (*e.g.*, Yu, 1999, Carbonell-Nicolau, 2010, 2015, and Scalzo, 2013). Yu (1999) allows for perturbed action spaces and payoff functions, but requires continuity of payoff functions. Carbonell-Nicolau (2010, 2015) and Scalzo (2013) allow for discontinuous payoffs but require fixed action spaces. In this paper we extend the results in Carbonell-Nicolau (2010) by allowing for perturbed payoffs and actions.

The notion of perturbed game used in this note differs from the definition adopted in Yu (1999). We argue in Section 2 that, in the presence of payoff discontinuities, perturbing actions and payoffs as in Yu (1999) poses problems. In fact, under Yu's approach it is easy to construct games whose perturbations do not include strategies that are of particular strategic significance to the players. Our discussion in Section 2 is framed in terms of a very simple example, which showcases the difficulties of the Yu approach and illustrates the intuitive appeal of the definition of a perturbed game proposed here.

2. Preliminaries

A ***normal-form game*** (or simply a ***game***) $G = (X_i, u_i)_{i=1}^N$ consists of a finite number N of players, a nonempty set of actions X_i for each player i, and a payoff function $u_i : X \to \mathbb{R}$ for each player i defined on the set of action profiles $X := \times_{j=1}^N X_j$.

For each player i, let X_i be a nonempty, compact, convex subset of a metric vector space. Let $X := \times_{i=1}^N X_i$ be endowed with the associated product topology. The sets $X_1, \ldots, X_N$ will be fixed throughout the analysis. Let $B(X)$ denote the set of bounded maps $f : X \to \mathbb{R}$. Let $K(X_i)$ denote the hyperspace of nonempty, compact, and convex subsets of X_i. Define

$$\boldsymbol{G}_X := \left(\times_{i=1}^N K(X_i)\right) \times B(X)^N.$$

A typical member of $\boldsymbol{G}_X$ is denoted $(Y, u) = (Y_1, \ldots, Y_N, u_1, \ldots, u_N)$ and can be viewed as a normal-form game $(Y_i, u_i|_{\times_{j=1}^N Y_j})_{i=1}^N$.

In Yu (1999), the space $B(X)^N$ is endowed with the metric $\gamma_X : B(X)^N \times B(X)^N \to \mathbb{R}$ defined by

$$\gamma_X\left((u_1, \ldots, u_N), (v_1, \ldots, v_N)\right) := \sum_{i \in N} \sup_{x \in X} |u_i(x) - v_i(x)|, \tag{1}$$

and, for each i, the space $K(X_i)$ is endowed with the Hausdorff metric topology. The associated product metric space $\boldsymbol{G}_X$, endowed with the corresponding product topology, constitutes the space of games considered in Yu (1999). This topology defines the notion of perturbed game used in Yu (1999), and we wish to argue here that this notion is not appropriate in the presence of payoff discontinuities. To illustrate, consider the one-person game $([0, 1], u)$, where $u(x) := 0$ if $x \in [0, 1)$ and $u(1) := 1$, and the sequence

☆ In memory of Nathan Wohl. Thanks to the anonymous referees and Rich McLean for valuable comments.

* Corresponding author.
E-mail address: carbonell-nicolau@rutgers.edu (O. Carbonell-Nicolau).

https://doi.org/10.1016/j.jmateco.2018.01.002
0304-4068/

$\big([0, 1-\frac{1}{n}], u\big)$, which converges to $([0, 1], u)$. Arguably, the strategy $x = 1$, which dominates every other strategy, is particularly important in this game, and it seems hard to justify an approximation that does not include this strategy or another strategy that plays a similar role. In particular, the games $([0, 1], u)$ and $\big([0, 1-\frac{1}{n}], u\big)$ appear markedly dissimilar, even for large n, and the sequence $\big([0, 1-\frac{1}{n}], u\big)$ does not seem to well-approximate $([0, 1], u)$.[1] By contrast, the sequence $\big([0, 1-\frac{1}{n}], v^n\big)$, where $v^n(x) := 0$ if $x \in [0, 1-\frac{1}{n})$ and $v^n(x) := 1-\frac{1}{n}$ if $x \in [1-\frac{1}{n}, 1]$ seems to better approximate $([0, 1], u)$ (for large n). Note that for the above topology, while the sequence $\big([0, 1-\frac{1}{n}], u\big)$ converges to $([0, 1], u)$, the sequence $\big([0, 1-\frac{1}{n}], v^n\big)$ does *not* converge to $([0, 1], u)$. In the next paragraph, we define a topology that is consistent with the idea that games of the form $\big([0, 1-\frac{1}{n}], v^n\big)$ are close to $([0, 1], u)$ (for large n) while games of the form $\big([0, 1-\frac{1}{n}], u\big)$ are not.[2]

Given i and $\{Y_i, Z_i\} \subseteq K(X_i)$, let $\mathcal{H}(Y_i, Z_i)$ be the set of all homeomorphisms h_i from Y_i to Z_i such that $h_i(A) \subseteq Z_i$ is convex if and only if $A \subseteq Y_i$ is convex. Let d_X be a compatible metric for X. Let $\mathfrak{G}_X$ represent the set of normal-form games $(Y_i, u_i|_{\times_{j=1}^N Y_j})_{i=1}^N$ such that $(Y, u) \in \boldsymbol{G}_X$. Note that a member of $\boldsymbol{G}_X$ uniquely determines a corresponding element of $\mathfrak{G}_X$, while there is a one-to-many mapping between $\mathfrak{G}_X$ and $\boldsymbol{G}_X$. For the members of $\mathfrak{G}_X$, we write $(Y_i, u_i|_{\times_{j=1}^N Y_j})_{i=1}^N$ and (Y, u) indistinctly, which entails a slight abuse of notation. Define the map $\alpha_X : \mathfrak{G}_X \times \mathfrak{G}_X \to \mathbb{R} \cup \{\infty\}$ by

$$\alpha_X((Y,u), (Z, v)) := \inf\Bigg\{\epsilon > 0 : \exists h \in \times_{i=1}^N \mathcal{H}(Y_i, Z_i) : \sum_{i=1}^N \sup_{x\in Y}|u_i(x) - v_i(h(x))| \le \epsilon \text{ and } \sup_{x\in Y} d_X(h(x), x) \le \epsilon\Bigg\},$$

if $\times_{i=1}^N \mathcal{H}(Y_i, Z_i) \neq \emptyset$, and $\alpha_X((Y, u), (Z, v)) := \infty$ if $\times_{i=1}^N \mathcal{H}(Y_i, Z_i) = \emptyset$. Now define the metric $\rho_X : \mathfrak{G}_X \times \mathfrak{G}_X \to \mathbb{R}$ by $\rho_X((Y, u), (Z, v)) := \min\{\alpha_X((Y, u), (Z, v)), 1\}$.[3] Throughout the sequel, we endow $\mathfrak{G}_X$ with the metric ρ_X.

Remark 1. As illustrated by the previous example, the metric ρ_X differs from the Yu metric. This discrepancy can even be found within the subdomain of continuous games. Indeed, for $X := [0, 1]$ and arbitrary u, the sequence of games $([0, \frac{1}{n}], u)$ in $\boldsymbol{G}_X$ converges to $(\{0\}, u)$ in the sense of Yu, and yet this sequence does not converge with respect to ρ_X in $\mathfrak{G}_X$ because none of its members is homeomorphic to the game $(\{0\}, u)$. Thus, convergence in the sense of Yu need not imply convergence with respect to ρ_X. The converse assertion is also true, as illustrated by the discontinuous game from the previous example.

Definition 1. A correspondence $\Phi : A \rightrightarrows B$ between topological spaces is ***upper hemicontinuous at*** $x \in A$ if the following condition is satisfied: for every neighborhood $V_{\Phi(x)}$ of $\Phi(x)$ there is a neighborhood V_x of x such that $y \in V_x$ implies $\Phi(y) \subseteq V_{\Phi(x)}$. Φ is ***upper hemicontinuous*** if it is upper hemicontinuous at every point in A.

Definition 2. A correspondence $\Phi : A \rightrightarrows B$ between topological spaces is ***lower hemicontinuous at*** $x \in A$ if the following condition is satisfied: for every open set $V \subseteq B$ with $V \cap \Phi(x) \neq \emptyset$ there is a neighborhood V_x of x such that $y \in V_x$ implies $\Phi(y) \cap V \neq \emptyset$. Φ is ***lower hemicontinuous*** if it is lower hemicontinuous at every point in A.

Definition 3. A strategy profile $x = (x_i, x_{-i})$ in X is a ***Nash equilibrium*** of $G = (X_i, u_i)_{i=1}^N$ if $u_i(y_i, x_{-i}) \le u_i(x)$ for every $y_i \in X_i$ and i.

One can define the Nash equilibrium correspondence as a set-valued map

$$\mathcal{E}_X : \mathfrak{G}_X \rightrightarrows X$$

that assigns to each game (Y, u) in $\mathfrak{G}_X$ the set of Nash equilibria of (Y, u), $\mathcal{E}_X(Y, u)$. Given a family of games $\mathfrak{G} \subseteq \mathfrak{G}_X$, the restriction of $\mathcal{E}_X$ to $\mathfrak{G}$ is denoted by $\mathcal{E}_X|_{\mathfrak{G}}$.

Definition 4. Given a class of games $\mathfrak{G} \subseteq \mathfrak{G}_X$, a Nash equilibrium x of $(Y, u) \in \mathfrak{G}$ is an ***essential equilibrium of*** (Y, u) ***relative to*** $\mathfrak{G}$ if for every neighborhood V_x of x there is a neighborhood $V_{(Y,u)}$ of (Y, u) such that for every $(Z, f) \in V_{(Y,u)} \cap \mathfrak{G}$, $V_x \cap \mathcal{E}_X(Z, f) \neq \emptyset$.

Definition 5. Suppose that $\mathfrak{G} \subseteq \mathfrak{G}_X$. A game (Y, u) in $\mathfrak{G}$ is ***essential relative to*** $\mathfrak{G}$ if every pure-strategy Nash equilibrium of (Y, u) is essential relative to $\mathfrak{G}$. When the domain of reference is clear from the context, we shall simply say that (Y, u) is an ***essential game***.

Remark 2. Suppose that $\mathfrak{G} \subseteq \mathfrak{G}_X$. A game (Y, u) in $\mathfrak{G}$ is essential relative to $\mathfrak{G}$ if and only if $\mathcal{E}_X|_{\mathfrak{G}}$ is lower hemicontinuous at (Y, u).

3. The results

The following definition was introduced in Barelli and Soza (2009).

---

[1] The idea that "good" approximations to an infinite discontinuous game should include strategies that are of particular strategic significance to the players is already discussed in Simon (1987) and Reny (2011) in the context of finite strategic approximations to infinite games.

[2] This is in fact an example in which a game with a dominant strategy can only be approximated, in the new topology, by games with a dominant strategy. This is obviously false about the Yu topology. We conjecture that this property holds in general, and we thank an anonymous referee for bringing up this point.

[3] It is easily seen that $\rho_X((Y, u), (Z, v)) = 0 \Leftrightarrow (Y, u) = (Z, v)$ for all (Y, u) and (Z, v) in $\mathfrak{G}_X$. Also, it is clearly the case that $\rho_X((Y, u), (Z, v)) = \rho_X((Z, v), (Y, u))$ for all (Y, u) and (Z, v) in $\mathfrak{G}_X$. To verify that the triangle inequality holds for ρ_X, fix (Y, u), (Y', u'), and (Y'', u'') in $\mathfrak{G}_X$ and note that given $h^1 \in \times_{i=1}^N \mathcal{H}(Y_i, Y_i')$ and $h^2 \in \times_{i=1}^N \mathcal{H}(Y_i', Y_i'')$,

$$\begin{aligned}\sum_{i=1}^N \sup_{x\in Y}|u_i(x) - u_i''(h^2(h^1(x)))| &= \sum_{i=1}^N \sup_{x\in Y}|u_i(x) - v_i''(x)| \\ &\le \sum_{i=1}^N \sup_{x\in Y}|u_i(x) - v_i'(x)| + \sum_{i=1}^N \sup_{x\in Y}|v_i'(x) - v_i''(x)| \\ &= \sum_{i=1}^N \sup_{x\in Y}|u_i(x) - u_i'(h^1(x))| + \sum_{i=1}^N \sup_{x\in Y}|u_i'(h^1(x)) - u_i''(h^2(h^1(x)))| \\ &= \sum_{i=1}^N \sup_{x\in Y}|u_i(x) - u_i'(h^1(x))| + \sum_{i=1}^N \sup_{x\in Y'}|u_i'(x) - u_i''(h^2(x))|,\end{aligned}$$

where $v_i' : Y \to Y'$ and $v_i'' : Y \to Y''$ are defined by

$$v_i'(x) := u_i'(h^1(x)) \quad \text{and} \quad v_i''(x) := u_i''(h^2(h^1(x))),$$

and

$$\begin{aligned}\sup_{x\in Y} d_X(h^2(h^1(x)), x) &\le \sup_{x\in Y} d_X(h^1(x), x) + \sup_{x\in Y} d_X(h^2(h^1(x)), h^1(x)) \\ &= \sup_{x\in Y} d_X(h^1(x), x) + \sup_{x\in Y'} d_X(h^2(x), x).\end{aligned}$$

Consequently, $\alpha_X((Y, u), (Y'', u'')) \le \alpha_X((Y, u), (Y', u')) + \alpha_X((Y', u'), (Y'', u''))$ and so $\rho_X((Y, u), (Y'', u'')) \le \rho_X((Y, u), (Y', u')) + \rho_X((Y', u'), (Y'', u''))$. Thus, ρ_X is indeed a metric on $\mathfrak{G}_X$.