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Sufficient conditions for weak reciprocal upper semi-continuity in mixed extensions of games

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ABSTRACT

We provide sufficient conditions for a game with discontinuous payoffs to be weakly reciprocally upper semi-continuous in mixed strategies. These conditions are imposed on the individual payoffs and not on their sum, and they can be readily verified in a large class of games even when the sum of payoffs in such games is not upper semi-continuous. We apply our result to establish the existence of mixed strategy equilibria in probabilistic voting competitions where candidates have very general utility functions as well as heterogeneous beliefs about the distribution of the voters.

1. Introduction

In a seminal paper, Reny (1999) identified two conditions, payoff security and reciprocal upper semi-continuity, which together play an important role in establishing the existence of Nash equilibria in games with discontinuous payoffs. Payoff security is a very weak notion of continuity that is imposed on the payoff of each player, and it implies the lower semi-continuity of the value function of each player (Prokopovych, 2011).[1] Reciprocal upper semi-continuity was initially discussed in Simon (1987) as a notion of complementary discontinuity, which intuitively says that if the payoff of an agent drops down in the limit of some sequence of strategies, then the payoff of another agent must jump up along the same sequence of strategies.

Unfortunately, these conditions can be difficult to verify in mixed strategies. Recent advances (see Monteiro and Page (2007) and Allison and Lepore (2015)) have allowed for straightforward verification of payoff security in mixed strategies in certain types of games. Furthermore, we know from Reny (1999) that reciprocal upper semi-continuity in mixed strategies implies reciprocal upper semi-continuity in pure strategies. The converse, however, is not true (see Example 5 in the appendix of Simon (1987)). The condition of reciprocal upper semi-continuity was relaxed in Bagh and Jofre (2006) where the notion of weak reciprocal upper semi-continuity was introduced.[2] However, the verification of weak reciprocal upper semi-continuity – and hence the verification of reciprocal upper semi-continuity – in mixed strategies remains a challenge, particularly when the sum of the payoffs is not upper semi-continuous.[3] This paper addresses this problem, providing simple – in a sense we will make precise shortly – sufficient conditions for weak reciprocal upper semi-continuity in mixed strategies. These conditions can be combined with the results in Monteiro and Page (2007). Allison and Lepore (2015) and Bagh and Jofre (2006) to establish the existence of a mixed strategy equilibrium in a large class of games with discontinuous payoffs.

Specifically, we consider a game G with a finite number of players with possibly discontinuous payoffs defined on a compact strategy set X. Let $\tilde{G}$ denote the extension of G to mixed strategies, and let Γ and $\tilde{\Gamma}$ respectively denote the graphs of G and $\tilde{G}$. The notion of better reply security of $\tilde{G}$ in mixed strategies (better reply security in pure strategies + quasi-concavity of G) introduced in Reny (1999) ensures that the set of Nash equilibria of $\tilde{G}$ (the set of pure strategy equilibria of G) is non-empty and closed in $\mathcal{M}(X)$, the space of probability measures on X (in X), and that limits of ε-equilibria – as ε goes to zero – are also equilibria. In order to establish that G and $\tilde{G}$ are better reply secure, one often

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[1] The value function of player i given the strategies of the other players, is the maximum payoff this player can obtain given the strategies of all the other players.

[2] Prokopovych (2011) modified weak reciprocal upper semi-continuity, in such a way that, in payoff secure games, the modified version becomes equivalent to better reply security.

[3] WRUSC in mixed strategies implies WRUSC in pure strategies. This observation and its proof in Appendix were provided by an anonymous referee.

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needs to establish that these games are weak reciprocal upper semi-continuous (henceforth WRUSC).[4] This is a condition that is imposed on the points that are in the topological frontier of the game consisting of the points that are in the closure of graph of the game but that are not on the graph itself. In other words, the frontier of $\tilde{\Gamma}$ is given by $Fr(\tilde{\Gamma}) = \text{cl}\,\tilde{\Gamma} \setminus \tilde{\Gamma}$ (resp. $Fr(\Gamma) = \text{cl}\,\Gamma \setminus \Gamma$ when only pure strategies are considered). On one hand, this condition is convenient because it is essentially ordinal in the sense that it is preserved under strictly increasing and continuous transformations, and it does not impose any requirement on the sum of the payoff functions as in Dasgupta and Maskin (1986) or on the aggregator function of the game as in Prokopovych and Yannelis (2014).[5] On the other hand, verifying this condition directly – particularly when the sum of the payoffs is not upper semi-continuous (henceforth usc) – can be rather complicated since it requires knowing the boundary set of $\tilde{\Gamma}$ in $\mathcal{M}(X) \times \mathbb{R}^I$.[6]

In this paper, we provide simple conditions that imply that $\tilde{G}$ is WRUSC. These conditions do not require computing $Fr(\tilde{\Gamma})$, and they can be verified even when the sum of the payoffs fails to be usc. These conditions are simple in the following sense: (i) they are imposed directly on the individual payoff functions of the game G, (ii) they are preserved under strictly increasing transformations that are continuous, (iii) they only require inspecting the behavior of the payoff functions of G over points of upper discontinuity i.e. points where some of the payoffs fail to be upper semi-continuous. The collection of such points – in many applications – form a very small set with a very simple structure. Our result is particularly easy to apply when the set of upper discontinuities for a player (again these are the points where his payoff fails to be upper semi-continuous) is the same for all the players. This covers a large class of games where the upper discontinuity in the payoff arises from a symmetric tie-breaking rule.

Finally, we apply our result to establish the better reply security of voting games with spatial competition. This, in turn, allows us to establish the existence (and the stability) of equilibria in such games under conditions that are considerably weaker than the current existence results in the literature on voting games.

Our main result, Theorem 1 in Section 3, is based on three observations: First, for every $(\mu^*, \alpha^*) \in Fr(\tilde{\Gamma})$, there exists some player j such that either the expected payoff of this player with respect to μ^* is strictly large than α_j^*, or μ^* must assign non-zero probability to the some points of discontinuity of the payoff of this player (Lemma 1 in Section 3). Second, the essential role of various notions of reciprocal upper semi-continuity is to relate the upper discontinuity points of individual payoff functions to the upper discontinuity points of the sum of the payoff functions (Lemma 2 in Section 3). Third, for any player with a given expected payoff, deviations in pure strategies can be expressed as deviations in mixed strategies with the same expected payoff.[7]

Alternatively, Theorem 4 in Prokopovych and Yannelis (2014) considers the better reply security of games whose payoffs can be transformed via positive affine functions into games with an upper semi-continuous sum of payoffs. As demonstrated in the examples in Prokopovych and Yannelis (2014), this approach is very useful in establishing the better reply security of many games. However, in some applications, the discontinuities arise from tie-breaking rules that are not completely specified. For such applications, establishing upper semi-continuity of the transformed payoffs may not be possible. Furthermore, our examples will show that our results can be applied even when no such transformations can be found (see Examples 1 and 2 in Section 3 and Example 3 in Appendix G).

2. Preliminaries

Consider a game with a finite number of players indexed by the set I. Each player has a strategy set X_i that is a compact subset of a metric space. Let $X = \prod_{i \in I} X_i$. Let $\mathcal{M}(X_i)$ be the space of Borel probability measures on X_i equipped with the standard weak topology, and let $\mathcal{M}(X) = \prod_{i \in I} \mathcal{M}(X_i)$. For every player i, the payoff is a Borel measurable bounded function $U_i : X \to \mathbb{R}$. The upper closure of U_i, denoted by $\overline{U_i}$, is defined as

$$\overline{U_i}(x) = \inf_{W \in \mathcal{W}(x)} \sup_{x' \in W} U_i(x'),$$

where $\mathcal{W}(x)$ is the collection of open sets in X containing x.

Recall the following properties of $\overline{U_i}$:

1. The function $\overline{U_i}$ is usc with $\overline{U_i} \geq U_i$. Moreover, for any $x \in X$, there exist $x_n \to x$ such that $\lim_n U_i(x_n) = \overline{U_i}(x)$.
2. If $U_i(x') \geq \alpha$ for all $x' \in W$ for some $W \in \mathcal{W}(x)$, then $\overline{U_i}(x) \geq \alpha$, which implies that if $\lim_n U_i(x_n) = \alpha$, for some $x_n \to x$, then $\overline{U_i}(x) \geq \alpha$.
3. If g is usc and $g(x) \geq U_i(x)$ for all $x \in X$, then $g(x) \geq \overline{U_i}(x)$ for all $x \in X$.

The function $\overline{U_i}$ is sometimes referred to as the *upper regularization* of the function U_i. Properties 1–3 above follow directly from the fact that $\text{hypo}\,\overline{U_i} = \text{cl}\,\text{hypo}\,U_i$ where $\text{hypo}\,U_i = \{(x, \alpha) \in X \times \mathbb{R} | U_i(x) \geq \alpha\}$ (see section D, Chapter 1 in Rockafellar and Wets (2009)).

Let D_i be the set of points in X where U_i is discontinuous. Since at any $x \in X$, $\overline{U_i}(x) \geq U_i(x)$, and $\overline{U_i}(x) = U_i(x)$ implies that U_i is upper semi-continuous at x, we will use D_i^{usc} to denote the set of points in X where U_i fails to be upper semi-continuous. In other words, we let D_i^{usc} be the set of points in $x \in X$ where $\overline{U_i}(x) > U_i(x)$. Clearly $D_i^{usc} \subseteq D_i$, and in many applications D_i^{usc} is actually much smaller than D_i. Finally, for every $x_i \in X_i$, we define $D_i(x_i) = \{x_{-i} \in X_{-i} | U_i(x_i, \cdot) \text{ is discontinuous in } x_{-i}\}$.

Given a game $G = (X_i, U_i, I)$, we denote by $\tilde{G} = (\mathcal{M}(X_i), EU_i, I)$ an extended game that is played on $\mathcal{M}(X)$, and the payoff of player i in the extended game is

$$EU_i(\mu_i, \mu_{-i}) = \int_{X_i} \int_{X_{-i}} U_i \, d\mu_{-i} d\mu_i.$$

The extended game $\tilde{G} = (\mathcal{M}(X_i), EU_i, I)$ has an equilibrium $\mu^* \in \mathcal{M}(X)$, if for every i,

$$EU_i(\mu_i^*, \mu_{-i}^*) \geq EU_i(\mu_i', \mu_{-i}^*), \ \ \forall \mu_i' \in \mathcal{M}(X_i).$$

Definition 1. $\tilde{G}$ is WRUSC, if for any $(\mu^*, \alpha^*) \in Fr(\tilde{\Gamma})$, there exists a player j and $\mu_j \in \mathcal{M}(X_j)$ such that

$$EU_j(\mu_j, \mu_{-j}^*) > \alpha_j^*.$$

3. The main result

We start with a lemma that clarifies the key property of the points in $Fr(\tilde{\Gamma})$.

Lemma 1. *If $(\mu^*, \alpha^*) \in Fr(\tilde{\Gamma})$, then there exists a player j such that either*

(a1) $\mu^*(D_j^{usc}) \neq 0$

or

(b1) $\alpha_j^* < \int U_j \, d\mu^*$.

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[4] The notion of WRUSC can be combined with the notion of payoff security to establish better reply security (see Bagh and Jofre (2006)).

[5] See the discussion on page 1030 in Reny (1999) on the significance of avoiding imposing condition on the sum of the payoffs in the game. For the proof that the property G is WRUSC is essentially ordinal, see the Appendix.

[6] If the sum of the payoffs is upper semi-continuous, then both G and $\tilde{G}$ are reciprocally upper semi-continuous, and therefore they are also WRUSC (see Reny (1999), page 1043).

[7] Mathematically speaking, this a consequence of Theorem C, Section 39 in Halmos (1974).