On aggregation and representative agent equilibria

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ABSTRACT

Aggregation is an often used tool in finance and macroeconomics, whereby economic equilibrium in a heterogeneous trader economy is characterized by means of the first order optimality conditions of a representative agent. In this paper we study the conditions under which a representative agent exists, and investigate the implications for the existence of equilibrium. The approach applies to markets which are incomplete, including markets with trading constraints, heterogeneous beliefs, and differential information.

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1. Introduction

Aggregation is an often used tool in finance and macroeconomics for characterizing an economic equilibrium in a heterogeneous trader economy; see Back (2010) and Ljungqvist and Sargent (2004). Aggregation implies the existence of a representative agent, whose first order conditions characterize aggregate consumption and the economy's state price density. This representation simplifies the characterization of equilibrium, thereby facilitating understanding and empirical estimation. It is well known that aggregation is possible in a heterogeneous trader economy when the equilibrium allocation is Pareto optimal, which occurs if the market is complete or when preferences satisfy linear risk tolerance (see Back (2010, Chapter 7) and Skiadas (2009, Chapter 3)). It is less well known, however, that the existence of a representative agent exists under a much weaker set of conditions. This occurs when the representative agent's aggregate utility function's weightings across the heterogeneous traders' utility functions are state dependent.

In dynamic stochastic economies, the representative agent paradigm has been an important construct for proving the existence of Radner (1972) equilibrium in complete markets; see Anderson and Raimondo (2008), Horst et al. (2010), Hugonnier et al. (2012), Riedel and Herzberg (2013), and Kramkov (2015). Perhaps more important, however, has been its use in proving the existence of Radner equilibrium in incomplete markets. In this regard, the representative agent approach with stochastic aggregate utility

function weights was first employed by Cuoco and He (1994), and later used by Basak and Cuoco (1998), Basak (2000), Hugonnier (2012), Hugonnier and Prieto (2015), Cheridito et al. (2015), and Jarrow (2017) to prove the existence of equilibrium in contexts where alternative approaches have not yet proven successful.

The purpose of this paper is twofold: (i) to study the conditions on a heterogeneous trader economy where a representative agent exists, and (ii) to prove a general equilibrium existence theorem for a heterogeneous trader economy using the existence of a representative agent equilibrium. This extended approach to the existence of a representative agent equilibrium applies to markets which are incomplete, including markets with trading constraints, heterogeneous beliefs, and differential information. We prove three key theorems in this paper. The first two theorems, in conjunction, provide a weak set of sufficient conditions for the existence of a representative agent equilibrium that reflects the aggregate demands and equilibrium price process of a heterogeneous trader economy. The third theorem gives a minimal set of sufficient conditions, together with the existence of a representative trader equilibrium, that imply the existence of an equilibrium in a heterogeneous trader economy. These theorems generalize earlier results in the literature mentioned in the preceding paragraph.

Let us briefly outline the approach and indicate why the proofs go through in the very general setting that we consider. The initial observation is that a numeraire invariance result can be proved under minimal conditions, in particular without any self-financing assumption and allowing for general markets and trading constraints; see Proposition 3.3. This suggests a natural notion of state price deflator, given in (3.4), which nests standard definitions. With this notion, a budget constraint inequality is easily obtained for any admissible strategy; see Proposition 3.7. Again, trading constraints

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are general, as are the cumulative endowments and dividends. This inequality turns out to be strong enough to prove that, given an equilibrium allocation, the aggregate consumption is optimal for the representative investor; see Theorem 4.8. A key observation is that the converse of Proposition 3.7 is never needed. Such a converse statement would assert that a strategy which satisfies the budget constraint is necessarily admissible, and would be difficult or impossible to prove at the required level of generality. The budget constraint inequality is also the crucial ingredient in the proofs of Theorems 4.9 and 4.10, which however also rely on results from convex and set-valued analysis.

An outline for this paper is as follows. Sections 2 and 3 present the model structure. Section 4 presents the first two theorems that provide sufficient conditions for the existence of a representative agent equilibrium, and explores existence of economic equilibrium in a heterogeneous trader economy. Section 5 concludes.

2. Preliminaries

Throughout this paper we fix a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, i.e. a filtered probability space whose filtration $\mathbb{F}$ is right-continuous; see Jacod and Shiryaev (2003, Definition I.1.2). The following standard notation will be used:

- The space of m-dimensional semimartingales is denoted $\mathscr{S}^m$, and we write $\mathscr{S} = \mathscr{S}^1$. We let $\mathscr{V}^+$ denote the set of all nondecreasing càdlàg adapted processes null at zero, and $\mathscr{V} = \mathscr{V}^+ - \mathscr{V}^+$ the set of all finite variation càdlàg adapted processes null at zero. Also, $\mathscr{A}$ ($\mathscr{A}_{\mathrm{loc}}$) is the set of processes in $\mathscr{V}$ of (locally) integrable variation.
- For $X \in \mathscr{S}^m$, $L(X)$ is the set of all predictable X-integrable processes $H = (H^1, \ldots, H^m)$. The stochastic integral is denoted $H \cdot X$. We always use vector stochastic integration as described in Shiryaev and Cherny (2002). In particular, $(H \cdot X)_0 = 0$ by definition.
- For $X \in \mathscr{S}^m$, $H \in L(X^1) \cap \cdots \cap L(X^m)$, we write

$$H \cdot X = (H \cdot X^1, \ldots, H \cdot X^m) \in \mathscr{S}^m.$$

For $Y \in \mathscr{S}$ and $H = (H^1, \ldots, H^m)$ with $H^i \in L(Y)$, $i = 1, \ldots, m$, we write

$$H \cdot Y = (H^1 \cdot Y, \ldots, H^m \cdot Y) \in \mathscr{S}^m.$$

For $X \in \mathscr{S}^m$ and $Y \in \mathscr{S}$ we write

$$[X, Y] = ([X^1, Y], \ldots, [X^m, Y]) \in \mathscr{S}^m.$$

For any $H \in L(X)$, Theorem 4.19 in Shiryaev and Cherny (2002) implies that

$$[H \cdot X, Y] = H \cdot [X, Y].$$

Throughout the paper we will use basic notions and results from convex analysis. The required material is reviewed in Appendix.

3. The financial market

We consider a financial market consisting of m assets whose prices at any time t are given by $S_t = (S_t^1, \ldots, S_t^m)$. The total net supply of asset i is denoted H_{tot}^i, which is assumed to be fixed over time. We write $H_{\mathrm{tot}} = (H_{\mathrm{tot}}^1, \ldots, H_{\mathrm{tot}}^m)$. The assets may pay dividends, and we let $D_t = (D_t^1, \ldots, D_t^m)$ denote the cumulative amounts that have been paid out by time t. It is assumed that $S \in \mathscr{S}^m$ and $D^i \in \mathscr{V}^+$ for each i. We allow $D^i \equiv 0$, in which case the associated asset does not pay any dividends at all. We do not require that a locally risk-free asset exists, but if it does, then it is one of the m assets mentioned above. Moreover, we do not assume that asset prices have been discounted (although we also do not rule it out.)

In this section we describe the trading strategies available in this environment, as well as state price deflators and the budget constraints that they induce.

3.1. Trading strategies

A *trading strategy* is a predictable process $H = (H^1, \ldots, H^m) \in L(S) \cap L(D)$, where H_t^i represents the number of units of asset i held at time t. The corresponding wealth process is

$$W = H^\top S = H^1 S^1 + \cdots + H^m S^m,$$

where W_t is simply the nominal value of the portfolio at time t.

The trading opportunities of any given trader depend on the information available to that trader, as well as on any investment restrictions the trader might face. This is captured by means of *trading constraints*, which associate to any given price process S a subset $\mathbf{K} = \mathbf{K}(S)$ of the set of all m-dimensional predictable processes. One then requires that the trader's trading strategy H belongs to $\mathbf{K}$.

Example 3.1 (*Trading Restrictions*). By choosing $\mathbf{K}$ suitably, a wide variety of trading constraints, as well as informational restrictions, can be enforced. For example:

- Solvency constraint: $\mathbf{K} = \mathbf{K}_{\mathrm{sol}} = \{H : H^\top S \geq 0\}$, enforcing a nonnegative portfolio value;
- Credit-line (or "admissibility") constraints: $\mathbf{K} = \mathbf{K}_{\mathrm{adm}} = \{H : H^\top S \geq -aL \text{ for some } a \geq 0\}$, where L is some nonnegative process. An example is $L = S^1$, where S^1 represents a locally risk-free bond;
- Short-sale prohibition: $\mathbf{K} = \{H : H^i \geq 0 \text{ for all } i\}$;
- Participation constraint: $\mathbf{K} = \{H : H^1 S^1 \leq (1 - \varepsilon) H^\top S\}$, the set of strategies that invest at most a fraction $1 - \varepsilon$ of the total portfolio value in asset 1;
- Informational constraints: $\mathbf{K} = \{H : H \text{ is } \mathbb{F}'\text{-predictable}\}$, where $\mathbb{F}' \subset \mathbb{F}$ is a given subfiltration that models the information available to the trader.

A trading strategy H is called $\mathbf{K}$-*feasible* if, up to indistinguishability,

$$\begin{cases} H \in \mathbf{K} \\ W = W_0 + H \cdot (S + D) + A \qquad (\text{some } A \in \mathscr{A}_{\mathrm{loc}}). \end{cases}$$

The presence of the process A means that the trading strategy is not self-financing in the usual sense, unless $A = 0$. Indeed, A represents the cumulative amount that must be added to (or withdrawn from) the gains process $W_0 + H \cdot (S + D)$ in order to match the portfolio value $W = H^\top S$. The set of all feasible trading strategies is denoted $\mathcal{F}(\mathbf{K})$; that is,

$$\mathcal{F}(\mathbf{K}) = \{H \in L(S) \cap L(D) : H \text{ is } \mathbf{K}\text{-feasible}\}.$$

Remark 3.2. By definition, a trading strategy H is S- and D-integrable relative to the filtration $\mathbb{F}$, and all stochastic integrals, such as $H \cdot (S + D)$, are computed relative to this filtration. In particular, this is the case also when the trading constraints $\mathbf{K}$ force H to be predictable relative to a subfiltration $\mathbb{F}' \subset \mathbb{F}$. This is important because it may happen that a process H is S-integrable relative to $\mathbb{F}'$, but $H \cdot S$ is not an $\mathbb{F}$-semimartingale; see Jeulin (1980, pages 46–47). To avoid difficulties in interpreting the trading outcome of a less informed trader, we require all trading strategies to be S-integrable relative to $\mathbb{F}$.