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Core and competitive equilibria: An approach from discrete convex analysis

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HIGHLIGHTS

- We extend the assignment market by utilizing discrete convex analysis.
- We consider the market in which buyers and sellers trade indivisible commodities for money.
- Buyers demand at most one unit of commodity, while sellers produce multiple units of several types of commodities.
- Sellers have $M^\natural$-convex cost functions.
- We prove that the core and the competitive equilibria exist and coincide.

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ABSTRACT

We extend the assignment market (Shapley and Shubik, 1972; Kaneko, 1976, 1982) by utilizing discrete convex analysis. We consider the market in which buyers and sellers trade indivisible commodities for money. Each buyer demands at most one unit of commodity. Each seller produces multiple units of several types of commodities. We make the quasi-linearity assumption on the sellers, but not on the buyers. We assume that the cost function of each seller is $M^\natural$-*convex*, which is a concept in discrete convex analysis. We prove that the core and the competitive equilibria exist and coincide in our market model.

1. Introduction

The assignment market (Shapley and Shubik, 1972; Kaneko, 1976, 1982) is a prominent model for the study of markets in which buyers and sellers trade indivisible commodities for money. The trade is bilateral, and buyers demand at most one unit of commodity. The main result of the assignment market is that the core and the competitive equilibria coincide. Compared with the case of continuous commodity space, the coincidence can be guaranteed without assuming infinite "copies" of agents. The result provides clear insight into the structure of the competitive equilibria.

We refer to previous works. Shapley and Shubik (1972) formulated the market and proved the existence of and the coincidence between the two concepts by using the duality theorem of the linear programming problem. Kaneko (1976) extended the model to cases where sellers can produce multiple units of several types of commodities. When sellers can produce multiple units, convexity of cost functions plays a crucial role. The difficulty here is how to define a "convex" function on a discrete domain. Kaneko (1976) assumed that the cost function of each seller is *separable convex*, i.e., the cost function is represented as the sum of univariate functions with non-decreasing marginal costs.[1] In the two studies mentioned above, the quasi-linearity assumption is imposed on the utility functions of buyers. This assumption implies no income effect, an assumption that is not suitable for a market where the proportion of price to income is not negligible. Therefore, Kaneko (1982) extended the previous models to cases without the quasi-linearity assumption. The model by Kaneko (1982) is called the generalized assignment market, abbreviated as the GAM model hereafter.

This study aims to extend the GAM model, in which sellers have separable convex cost functions. This assumption is restrictive in the following sense: if a seller has a separable convex function, then the cost of producing one commodity is always independent of the

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[1] Here, the "marginal cost" means the cost of producing one more unit.

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production of other types of commodities. To relax this assumption, we describe a convexity assumption by using a concept in discrete convex analysis (Murota, 2003). We assume that the cost function of each seller is M$^\natural$-*convex*. The separable convex function in previous studies is a special case of the M$^\natural$-convex function. In addition, the M$^\natural$-convex function allows the cost of producing a commodity to be dependent on the production of other types of commodities.

The M$^\natural$-convex function has been applied to market theory or matching theory in the literature. Murota (2003) proved the existence of a competitive equilibrium in the market where agents trade indivisible commodities for money and have M$^\natural$-concave utility functions. Fujishige and Tamura (2007) provided a generalization of the assignment market (Shapley and Shubik, 1972) and the marriage market (Gale and Shapley, 1962) by utilizing discrete convex analysis. Kojima et al. (2015) applied discrete convex analysis to two-sided matching markets in which certain distributional constraints exist. They proved that if the preferences of hospitals can be represented as an M$^\natural$-concave function, then the generalized deferred acceptance mechanism is strategy-proof and yields a stable matching.

Our main contribution is to provide a new application of the M$^\natural$-convex function to market theory. Consider the market in which buyers demand at most one unit of commodity and sellers have M$^\natural$-convex cost functions. In this market, under some conditions, the core and the competitive equilibria exist and coincide. By using the technique of discrete convex analysis, we also prove that if buyers have quasi-linear utility functions, the set of competitive price vectors is a lattice.

The remaining part of this study is organized as follows. In Section 2, we introduce the market model. In Section 3, we define the M$^\natural$-convex function and refer to previous studies on the relationship between the gross-substitutes condition and M$^\natural$-concavity. Section 4 presents the main results. Section 5 concludes this paper. All proofs are provided in Section 6.

2. Market model

2.1. Buyers and sellers

Let H denote the set of *buyers*, J the set of *sellers*, and L the set of *commodities*. The three sets are non-empty and finite.

For each $l \in L$, let $\mathbf{1}_l$ denote the lth unit vector, and $\mathbf{1}_0$ denote the 0-vector. We define

$$X = \left\{\mathbf{1}_l : l \in L \cup \{0\}\right\}.$$

We define the *consumption set* by $X \times \mathbb{R}_+$. An element $(\mathbf{1}_l, c) \in X \times \mathbb{R}_+$ means that a buyer consumes one unit of commodity l and c amount of money. Let $I^h \geq 0$ denote the *income* of buyer $h \in H$. For each $h \in H$ and $p \in \mathbb{R}^L_+$, we define

$$X^h_p = \{x \in X : p \cdot x \leq I^h\}.$$

In words, X^h_p is the set of commodities that h can consume at price vector p.

A buyer $h \in H$ has a utility function $U^h : X \times \mathbb{R}_+ \to \mathbb{R}$. We make the following assumptions:

A1 (*Monotonicity and continuity*). $U^h(\cdot, \cdot)$ is strictly monotonic and continuous with respect to the second argument.

A2 (*Indispensability of money*). For each $x \in X$, $U^h(\mathbf{0}, I^h) \geq U^h(x, 0)$.

For each $h \in H$, we define the *demand correspondence* $D^h : \mathbb{R}^L_+ \twoheadrightarrow X$ by

$$D^h(p) = \arg\max_{x \in X^h_p} U^h(x, I^h - p \cdot x) \quad \text{for all } p \in \mathbb{R}^L_+.$$

A seller $j \in J$ has a cost function $c^j : \mathbb{Z}^L_+ \to \mathbb{R} \cup \{+\infty\}$. Here, $c^j(x) = +\infty$ means that j cannot produce x. Let $Y^j = \{x \in \mathbb{Z}^L_+ : c^j(x) < +\infty\}$. We assume that $c^j(\mathbf{0}) = 0$ and Y^j is bounded. We also assume the following:

$$\text{for each } l \in L, \quad \text{there exists } j \in J \text{ such that } \mathbf{1}_l \in Y^j. \tag{1}$$

This means that for each commodity, there is at least one seller who can produce the commodity.

For two vectors $x, y \in \mathbb{Z}^L$, $x \geq y$ means that $x_l \geq y_l$ for all $l \in L$. We make the following assumption:

A3 (*Monotonicity*). $c^j(\cdot)$ is monotone-nondecreasing, i.e., for any $x, y \in Y^j$, $x \geq y$ implies $c^j(x) \geq c^j(y)$.

For each $j \in J$, we define the *supply correspondence* $S^j : \mathbb{R}^L_+ \twoheadrightarrow Y^j$ by

$$S^j(p) = \arg\max_{x \in Y^j}\{p \cdot x - c^j(x)\} \quad \text{for all } p \in \mathbb{R}^L_+.$$

In Section 3, we additionally make a convexity assumption on $c^j(\cdot)$.

2.2. Sellers with initial endowments

In Section 2.1, we assumed that sellers have cost functions and produce commodities. We briefly explain that our market model subsumes the case in which sellers initially own indivisible commodities.

We consider the market in which buyers are defined in the same way as in Section 2.1, while sellers have initial endowments of indivisible commodities. Let $e^j \in \mathbb{Z}^L_+$ denote the initial endowment of $j \in J$. For each $j \in J$, we define

$$\bar{Y}^j = \{x \in \mathbb{Z}^L_+ : 0 \leq x_l \leq e^j_l \text{ for all } l \in L\}.$$

We assume that each seller j has a quasi-linear utility function $U^j : \mathbb{Z}^L_+ \times \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$. Namely, there is a valuation function $u^j : \mathbb{Z}^L_+ \to \mathbb{R} \cup \{-\infty\}$ such that

$$U^j(x, c) = u^j(x) + c \quad \text{for all } (x, c) \in \mathbb{Z}^L_+ \times \mathbb{R}_+.$$

We assume that, for each $x \in \mathbb{Z}^L_+$, $u^j(x) \in \mathbb{R}$ if $x \in \bar{Y}^j$ and $u^j(x) = -\infty$ otherwise. This assumption means that sellers do not buy any indivisible commodities. We define the demand correspondence $D^j : \mathbb{R}^L_+ \twoheadrightarrow \bar{Y}^j$ by

$$D^j(p) = \arg\max_{x \in \bar{Y}^j}\{u^j(x) + p \cdot (e^j - x)\}. \tag{2}$$

If $x \in D^j(p)$, this means that it is optimal for j to sell the commodities $e^j - x$ at price vector p.

We can embed the above market into the market in Section 2.1. For each seller $j \in J$, we define the cost function $c^j : \mathbb{Z}^L_+ \to \mathbb{R} \cup \{+\infty\}$ by

$$c^j(x) = u^j(e^j) - u^j(e^j - x) \quad \text{for all } x \in \mathbb{Z}^L_+. \tag{3}$$

For each $x \in \bar{Y}^j$, $c^j(x)$ represents a decrease in utility by giving $e^j - x$ to other agents. One can verify that $S^j(p) = \{e^j - x : x \in D^j(p)\}$ for all $p \in \mathbb{R}^L_+$. Thus, the seller with valuation function $u^j(\cdot)$ supplies the same amount of commodities as the seller with cost function $c^j(\cdot)$ does.

2.3. Core and competitive equilibrium

Define $N = H \cup J$. A *coalition* is a subset S of N such that $S \cap H \neq \emptyset$, $S \cap J \neq \emptyset$. For each coalition S, a tuple $(x^i, t^i)_{i \in S}$ is called an *S-allocation* iff

$$(x^h, I^h - t^h) \in X \times \mathbb{R}_+ \quad \text{for all } h \in S \cap H,$$

$$(x^j, t^j) \in Y^j \times \mathbb{R}_+ \quad \text{for all } j \in S \cap J,$$

$$\sum_{h \in S \cap H} (x^h, t^h) = \sum_{j \in S \cap J} (x^j, t^j).$$