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Strategic sharing of a costly network

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ABSTRACT

We study minimum cost spanning tree problems for a set of users connected to a source. Prim's algorithm provides a way of finding the minimum cost tree m. This has led to several definitions in the literature, regarding how to distribute the cost. These rules propose different cost allocations, which can be understood as compensations and/or payments between players, with respect to the *status quo point*: each user pays for the connection she uses to be linked to the source. In this paper we analyze the rationale behind a distribution of the minimum cost by defining an *a priori* transfer structure. Our first result states the existence of a transfer structure such that no user is willing to choose a different tree from the minimum cost tree. Moreover, given a transfer structure, we implement the above solution as a subgame perfect equilibrium outcome of a game where players decide sequentially with whom to connect. Finally, we obtain that the existence of a transfer structure supporting an allocation characterizes the core of the monotone cooperative game associated with a minimum cost spanning tree problem. This transfer structure is called *social transfer structure*. Therefore, the minimum cost spanning tree emerges as both a social and individual solution.

1. Introduction

Many real life situations may be understood as network-related issues in which agents are interested in working towards the goal of a common project and are responsible for paying the total cost of its implementation. For instance, if an application needs to send large amounts of the same information or goods to multiple devices (e.g., streaming video, voice-conferencing or software distribution applications), multicast is an important means of transmitting this information. Another example is an irrigation system that supplies water to irrigated land from a water dam. This structure is a network of permanent and temporary conduits (canals and pipes). All of these matters are related to the cost sharing that agents face depending on the implemented structure (*i.e.*, the chosen multicast, the connected canals and pipes).

This paper deals with how the cost of any tree should be shared so that users are better off by implementing the minimum cost tree. We construct a *scheme of transfers* such that under a non-cooperative approach, the minimum cost tree is the unique outcome of a subgame perfect equilibrium of a sequential game; and under a cooperative perspective, the sharing of the minimum cost tree is in the core of a cooperative game associated to the network. Specifically, departing from a social planner solution, this paper provides a policy such that each agent will pay less if they implement the minimal cost tree.

We focus on a complete network with a finite number of users and an additional special node, the *source*. Users located at different points want to be connected to the source either directly or indirectly through other users. There is a cost of building a link between two agents, and from each agent to the source. Prim's algorithm (Prim, 1957) offers a way to connect all users to the source, where the total cost of all the edges in the tree is minimized. This solution, called the *minimum spanning tree*, is appealing from an economic point of view, because the algorithm provides an optimal (social) solution: if a social planner had to select a network, she would pick the one involving the lowest cost for society. Nevertheless, if every user could choose the agent with whom to connect by paying just the link she uses, she might find it profitable to secede from the socially optimal solution.[1]

The tension between the social and individual maximization problem opens a point at issue: could the social planner provide

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[1] Real world situations suggest that agents do not necessarily choose the social optimum (see Bergantiños and Lorenzo, 2004, for an example).

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a way[2] to allocate the cost of implementing any tree in such a way agents agree to implement only the minimal cost tree? To examine this matter in question, consider an allocation of the minimum cost proposed by the social planner. It is natural to think that no user is willing to pay more than her direct cost to the source. Hence, the difference between this cost and the proposed allocation, can be understood as the profit that an user obtains from cooperation in building the minimum cost spanning tree. On the other hand, once the minimum cost spanning tree is constructed, each user knows the cost of the (direct) link she uses to be connected. Then, the difference between the proposed allocation and the cost of the respective links, positive or negative, denotes the amount she must pay or is granted respectively[3] to accept implementing the social optimum. Therefore, implicitly, a number of *transfers* appears *a posteriori* associated to any cost allocation.

Following this idea, the social planner may enforce the choice by the users of a specific tree through the *a priori* establishment of some transfers. The collection of transfers for each user represents the amount that she pays to the individual with whom she connects *via* a tree. We call this a *transfer structure*. In addition, the way of allocating the cost of a tree will depend on the transfers getting the final cost for each user as the sum of the following three terms:

(i) the *direct cost*, whereby each user pays the cost of the link she uses;
(ii) the *user transfer*, which corresponds to the amount the user pays to the individual to whom she is connected. This term is understood as the amount paid for being linked to other users to implement the tree; and
(iii) the quantity she receives from other users to properly connect with her, called the *user grant*.

Our first result states the existence of a family of transfers such that any individual favors the minimum cost spanning tree. Moreover, our proof is constructive. We provide the procedure to set the user transfers and grants allowing the social planner to induce the desirable incentive to the users. We call this particular scheme as a *social transfer structure*. Notice the required condition of the structure. For instance, the well known Bird's proposal (Bird, 1976) of sharing the cost consists of each agent paying the cost of the link she uses to be connected. If we apply such a sharing, there are no transfers among agents and the minimum cost spanning tree may not be the preferred tree for any agent[4]. If user grant exceeds, individual will prefer the direct link to the source. If user transfers are not large enough, individual will still choose the link with the lowest cost. Therefore, to establish the right amount as transfer and grants becomes a subtle puzzle. Examples of this sort of policies are the subsidies in water network and tolls for roads. (See policies of the World Bank.[5])

Our second result stems from strategic issues: how agents agree to implement the minimum cost spanning tree. As users are motivated by self-interest then any solution should be in tune with incentive-compatible behavior. The rule of how to share the cost establishes the incentive structure and hence the associated equilibrium. Consider the sequential game where players decide with whom to connect, and payoff vectors are computed by the sum of the direct cost, the user transfer and the user grants. The minimum cost spanning tree will be the outcome of a subgame perfect equilibrium of this sequential game. Notice that the way to share the total cost is to relate the proposal to the whole network structure, since not only the transfers made for the minimum cost spanning tree are important but the transfers outside of this tree also matter. In order to sustain a tree it is necessary to be aware of any possible tree that may generate a profitable deviation of a single user. The effect of the transfers is also to break off any deviation for any tree and for any user. Moreover for any social transfer structure, the path associated to the minimum cost spanning tree is the unique outcome of a subgame perfect equilibrium that survives for any possible sequence in the order in which users act. Consequently, the individual choices coincide with the social solution.

Finally, our third result proves that, for any social transfer structure, the allocation it provides is in the core of the monotone cooperative game associated to a minimum cost spanning tree problem. In fact, the set of allocations provided by a social transfer structures coincides with the core of this cooperative game.

1.1. Context within the literature

The minimum cost spanning tree (*mcst*, hereafter) problem is the most studied model in the axiomatic literature on combinatorial optimization games.[6] In his seminal paper, Bird (1976) proved that the core of the *mcst* problem is non-empty and provided an intuitive core selection which is easy to compute. Nevertheless, this solution is not continuous and small changes in the cost matrix can lead to very different allocations. Moreover, Bird's solution is an extreme point in the core that is systematically unfair to some players: an agent close to the source pays her (possibly) costly connection (the maximum amount for her, consistent with the core constraints), while other agents pay the minimum (see Example 1 in the next section).

Other solutions have been defined for allocating the total cost in a *mcst* problem, mainly based on the Shapley value[7] of a cooperative game (see for instance Kar, 2002; Dutta and Kar, 2004). The *folk solution*, "the most famous cost allocation for *mcst* problems" (Trudeau, 2012), first suggested by Feltkamp et al. (1994) and rediscovered independently by Bergantiños and Vidal-Puga (2007), is obtained as the Shapley value of a cooperative game defined by an *irreducible* cost matrix constructed from the original one. This solution is a core selection and satisfies many appealing properties (see Bergantiños and Vidal-Puga, 2007).

Some literature analyzes the *mcst* problem from a strategic perspective. In particular, Bergantiños and Lorenzo (2004) and Bergantiños and Lorenzo (2005) proposed a noncooperative game in which agents decide, simultaneously, whether or not they want to be connected to the source. They assumed that agents want to be connected, with a penalty if an agent is ultimately not connected. They characterized the subgame perfect equilibria (SPE) of this game, which support Bird's solution. More recently, Bergantiños and Vidal-Puga (2010) defined a bargaining mechanism whose SPE support the folk solution. In this mechanism, agents simultaneously make an offer to decide if the tree will be built.

Our strategic approach differs from the aforementioned ones. First, we fix a transfer structure and a random order of choices for

2 A common approach in the literature consists of the design of rules that satisfy axioms representing fair properties commonly accepted by society (see, for instance, Bergantiños and Vidal-Puga, 2007; Bergantiños and Vidal-Puga, 2010; Bergantiños and Lorenzo-Freire, 2008; Bogomolnaia and Moulin, 2010; Trudeau, 2012).

3 Notice that the sum of these grants and payments equals zero.

4 When any agent uses the link with lower cost and indeed implements the minimum cost spanning tree then it is not necessary any transfers or grants.

5 It was retrieved on January 2016 from: http://www.worldbank.org/transport/roads/toll_rds.htm; http://water.worldbank.org/shw-resource-guide/finance/sanitation-subsidies/subsidy-mechanisms.

6 As pointed out by Bogomolnaia and Moulin (2010).

7 An exception is the paper by Granot and Huberman (1984), in which the solution is based on the nucleolus of the cooperative game.