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Conditional preference orders and their numerical representations

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ABSTRACT

We provide an axiomatic system modeling conditional preference orders which is based on conditional set theory. Conditional numerical representations are introduced, and a conditional version of the theorems of Debreu on the existence of numerical representations is proved. The conditionally continuous representations follow from a conditional version of Debreu's Gap Lemma the proof of which relies on a conditional version of the axiom of choice, free of any measurable selection argument. We give a conditional version of the von Neumann and Morgenstern representation as well as automatic conditional continuity results, and illustrate them by examples.

1. Introduction

In decision theory, the normative framework of preference ordering classically requires the completeness axiom. Yet, there are good reasons to question completeness as famously pointed out by Aumann (1962):

> Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. [...] For example, certain decisions that an individual is asked to make might involve highly hypothetical situations, which he will never face in real life. He might feel that he cannot reach an "honest" decision in such cases. Other decision problems might be extremely complex, too complex for intuitive "insight", and our individual might prefer to make no decision at all in these problems. Is it "rational" to force decision in such cases?

Aumann's remark, supported by empirical evidence, triggered intensive research in terms of interpretation, axiomatization and representation of general incomplete preferences, see Richter (1966), Peleg (1970), Bewley (2001), Dubra and Ok (2002), Dubra et al. (2004), Eliaz and Ok (2006), Evren and Ok (2011) and the references therein. These authors consider incompleteness either as a result of status quo, see Bewley (2001), or procedural decision making, see Dubra and Ok (2002), and the numerical representations are in terms of multi-utilities. However, Aumann's quote and a correspondence with Savage (Robert, 1987), where he exposes the idea of state-dependent preferences, suggest that the lack of information underlying a decision making is a natural source of incompleteness. For instance, consider the simple situation where a person has to decide between visiting a museum or going for a walk on Sunday in one month from now. She cannot express an unequivocal preference between these two prospective situations since it depends on the knowledge of uncertain factors like the weather, availability of an accompanying person, etc. This information-based incompleteness suggests a contingent form of completeness. For instance, conditioned on the event "sunny and warm day" she prefers a walk. In this way, a complex decision problem, provided sufficient information, leads to an "honest" decision. The present work suggests a framework formalizing this idea of a contingent decision making and its quantification.

Numerous quantification instruments in finance and economics entail a conditional dimension by mapping prospective outcomes to random variables such as for instance conditional and dynamic monetary risk measures (Detlefsen and Scandolo, 2005; Cheridito et al., 2006; Cheridito and Kupper, 2009; Acciaio et al., 2012), conditional expected utilities and certainty equivalents, dynamic assessment indices (Fritelli and Maggis, 2011; Bielecki et al., 2016) or recursive utilities (Epstein and Zin, 1989; Duffie and Epstein, 1992). However, few papers address the axiomatization of conditional preferences underlying these conditional quantitative instruments. In this direction is the work of Luce and Krantz (1971) where an event-dependent preference ordering is considered and studied. Their approach is further refined and extended in Wakker

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(1987) and Karni (1993a,b). State-wise dependency is used in Kreps and Porteus (1978, 1979) and Maccheroni et al. (2006) to study intertemporal preferences and a dynamic version of preferences, respectively. Remarkable is the abstract approach by Skiadas (1997a,b). He provides a set of axioms modeling conditional preferences on random variables which admit a conditional Savage representation of the form

$$U(x) = E_Q[u(x) \mid \mathcal{A}]$$

where $\mathcal{A}$ is an algebra of events representing the information, Q is a subjective probability measure and u is a utility index. As in the previous works, its decision-theoretical foundation consists of a whole family of total pre-orders $\succcurlyeq^A$, one for each event $A \in \mathcal{A}$, and a consistent aggregation property in order to obtain the conditional representation. However, the decision maker is assumed to implicitly take into account a large number[1] of complete pre-orders.

Our axiomatic approach differs in so far as it considers a *single* but possibly *incomplete* preference order $\succcurlyeq$ instead of a whole family of complete preference orders. Even if one cannot a priori decide whether $x \succcurlyeq y$ or $y \succcurlyeq x$ for any two prospective outcomes, or acts, there may exist a contingent information A conditioned on which x is preferable to y. In this case we formally write $x|A \succcurlyeq y|A$. The set of contingent information is modeled as an algebra of events $\mathcal{A} = (\mathcal{A}, \cap, \cup, {}^c, \emptyset, \Omega)$ of a state space Ω.[2] In order to describe the conditional nature of the preference, we require that $\succcurlyeq$ interacts consistently with the information, that is,

- *consistency:* if $x|A \succcurlyeq y|A$ and $B \subseteq A$, then $x|B \succcurlyeq y|B$;
- *stability:* if $x|A \succcurlyeq y|A$ and $x|B \succcurlyeq y|B$, then $x|A \cup B \succcurlyeq y|A \cup B$;
- *local completeness:* for every two acts x and y there exists a non-empty event A such that either $x|A \succcurlyeq y|A$ or $x|A \succcurlyeq y|A$.

These assumptions bear a certain normative appeal in view of the conditional approach that we are aiming at. In the context of the previous example, consistency says that if the person prefers a walk over a visit to a museum whenever it is "sunny" or "warm", then a fortiori she prefers a walk if it is "sunny". Stability tells that if she prefers a walk whenever it is "sunny" or "rainy", then on any day where at least one of these conditions is met she will go for a walk. In contrast to classical preferences, we only assume a local completeness: For any two situations she is able to meet a decision provided enough – possibly extremely precise[3] – information. In our example, there exists a rather unlikely, but still non-trivial, coincidence of the conditions *'sunny'*, *'humidity between 15 and 20%'* and *'wind between 0 and 10km/h'* under which she prefers a walk to the museum. Unlike classical completeness, the information necessary to decide between two acts x and y depends on the pair (x, y). Note that if the set of contingent information reduces to the trivial information $\mathcal{A} = \{\emptyset, \Omega\}$, then, as expected, a conditional preference is a classical complete preference order. In particular, classical decision theory is a special case of the conditional one.

Observe that our approach, as Luce and Krantz (1971), Wakker (1987), Kreps and Porteus (1978) and Skiadas (1997a,b), considers an *exogenously* given set of information or events as the source of incomplete decision making. Whereas in Bewley (2001), Dubra and Ok (2002) and the related subsequent literature on incomplete preference, the incompleteness and the resulting multi-valued representations yield an endogenous information about the nature of the incompleteness. Incompleteness there is however not in terms of an algebra of events, and therefore not specifically related to a contingent decision making.

Our approach is also not a priori *dynamic* in the sense that a single algebra of available information is given for the contingent decision making. We do not address the question of progressive learning over time as new information reveals, resulting in an update of decisions. This incremental learning approach in decision making is investigated by Kreps and Porteus (1978), and recently by Dillenberger et al. (2014) as well as Piermont et al. (2015).[4] In these articles, the agent learns over time and may modify her behavior according to the new information as well as her previous choice making. However, the underlying information structure is exogenously given — either by a fixed dynamic structure by means of a filtration or a random tree, or by the filtration generated by the consumption paths, or even by the filtration generated by the previous preference orders. Our approach may help in these cases by considering a sequence of conditional preference orders $\succcurlyeq_0, \succcurlyeq_1, \ldots, \succcurlyeq_t, \ldots$ with respect to an increasing sequence of algebra of events $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_t \subseteq \cdots$ each of which for every point in time. We can provide an axiomatic system to describe these conditional preference orders $\succcurlyeq_t$ for each given time t, and derive a sequence of conditional numerical representations U_t. Since we only address the case of a single information structure, that is, at a fixed given time t, we intentionally left out the following two questions in the dynamic context. First, whether the decision making at time t is influenced by the past information, that is a Markovian versus non-Markovian decision making. Second, the impact at time t of past and eventually future decisions. In other terms, the interdependence structure over time of these preferences and the consequences for the dynamic utility representation[5] in terms of time consistency.[6]

Although being intuitive, it is mathematically not obvious what is meant by a contingent prospective act $x|A$. The formalization of which corresponds to the notion of a conditional set, introduced recently by Drapeau et al. (2016). An heuristic introduction to conditional sets is given in Section 2. For an exhaustive mathematical presentation we refer to Drapeau et al. (2016). The formalization and properties of conditional preferences are given in Section 3. In Section 4, we address the notion of conditional numerical representation and prove a conditional version of Debreu's existence result of continuous numerical representations. While the proof technique differs, the classical statements in decision theory translate into the conditional framework. For instance, a conditional version of the classical representation of von Neumann and Morgenstern (1947) is presented in Section 5. The representation of Debreu requires topological assumptions that often are not met in practice. In Section 6, we provide conditional results that allow to extend Debreu and Rader's theorem in a more general framework, and present automatic continuity results which allow to bypass topological assumptions. We illustrate each of these cases by examples. These results in their classical form rely on the Gap Lemma of Debreu (1954, 1964) the conditional adaptation of which does not involve any measurable selection arguments but derives from a conditional version of the axiom of choice. Section 7 is dedicated to the formulation and the proof of this conditional Gap Lemma. In the Appendix, we gather some technical results and most of the proofs.

1 In a five steps binary tree, 4.294.967.296 is the cardinality of the family of total pre-orders $\succcurlyeq^A$.

2 Conditional set theory (Drapeau et al., 2016) allows the contingent information to be any complete Boolean algebra.

3 Indeed, the smaller the event, the more precise in which state of the world this event may occur. The most precise event being the singleton.

4 Though, Dillenberger et al. (2014) consider a static approach resulting in dynamic utility valuations that are deterministic.

5 For instance, time consistency, Bellman principle, weaker time consistency, etc.

6 A topic of intensive study in mathematical finance, see Cheridito et al. (2006), Cheridito and Kupper (2009), Acciaio et al. (2012), Cialenco et al. (2014), Bielecki et al. (2016) among others.