Journal of Mathematical Economics 63 (2016) 164–173

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

A theory of stochastic choice under uncertainty

Edi Karni [a,b,1], Zvi Safra [b,c,*]

[a] Department of Economics, Johns Hopkins University, United States
[b] Warwick Business School, University of Warwick, United Kingdom
[c] Emeritus, Tel Aviv University, Israel

ARTICLE INFO

Article history:
Received 21 January 2016
Received in revised form
2 February 2016
Accepted 4 February 2016
Available online 26 February 2016

Keywords:
Random choice
Decision making under uncertainty
Incomplete preferences
Preferences over menus
Subjective state space

ABSTRACT

In this paper we propose a characterization of stochastic choice under risk and under uncertainty. We presume that decision makers' actual choices are governed by randomly selected states of mind, and study the representation of decision makers' perceptions of the stochastic process underlying the selection of their state of mind. The connections of this work to the literatures on random choice, choice behavior when preference are incomplete; choice of menus; and grades of indecisiveness are also discussed.

1. Introduction

In this paper, we develop a theory of random choice under uncertainty and under risk motivated by the recognition that there are situations in which the decision maker's tastes are subject to random variations. In these situations, a decision maker's choice behavior displays a stochastic pattern represented by a probability distribution on the set of alternatives.

The idea advanced in this paper is that variability in choice behavior is an expression of internal conflict among distinct inclinations, or distinct "selves" of the decision maker, whose assessments of the alternatives are different. We refer to these inclinations as "states of minds" and assume that, analogous to a state of nature, a state of mind resolves the uncertainty surrounding a decision maker's true subjective beliefs and/or tastes. Our theory presumes that, at a meta level, decision makers entertain beliefs about their likely state of mind when having to choose among uncertain, or risky, prospects; that their actual choice is determined by the state of mind that obtains; and that the observed choice probabilities are consistent with these beliefs. In other words, a decision maker's state of mind governs his choice behavior in the sense that, when having to choose among acts (or lotteries), a state of mind, encompassing beliefs and risk attitudes, is selected at random and that state of mind determines which alternative is chosen. The focus of our investigation is *the representation of the decision maker's perception of the stochastic process underlying the selection of his state of mind*. We presume that this process is accessible by introspection and that it agrees with the empirical distribution characterizing the random choice rule.

The fact that states of mind are preference relations has two crucial implications: It renders the evaluation of the outcomes – acts or lotteries, as the case may be – dependent on the state (of mind) and it lends the states of mind the inherently quality of private information (as opposed to states of nature which are observable). These implications raise two difficulties. First, because the preference relation is state dependent, subjective expected utility theory fails to deliver a unique prior. Second, because states of mind are private information, they express themselves, indirectly, through choices among menus rather than directly through the choice of acts. To overcome the first difficulty, we apply a modified version of the model of Karni and Schmeidler (1980, 2016). To overcome the second difficulty, building on ideas introduced by Kreps (1979) and developed by Dekel et al. (2001), we derive preferences over acts from those on menus. Hence, we assume that a decision maker is characterized by two primitive preference relations: a preference relation on the set of menus of alternatives depicting his actual choice behavior and an introspective preference relation on hypothetical mental state-act lotteries.

The preference relation on the set of menus induces preferences on the set of mental acts (that is, mappings from the set of

* Corresponding author at: Warwick Business School, University of Warwick, United Kingdom.

1 Part of this work was done during my visit to EIEF, Rome.

http://dx.doi.org/10.1016/j.jmateco.2016.02.001
0304-4068/

states of mind to the set of uncertain, or risky, prospects). Both the preference relation on the set of mental acts and that on the mental state-act lotteries are assumed to satisfy the von Neumann–Morgenstern axioms and, when a natural correspondence connects between their domains, they are required to agree with each other. This model yields a representation of the preference relations over mental acts induced by menus that takes the form of subjective expected utility with state-dependent utility functions defined on uncertain, or risky, prospects and a unique subjective prior on the set of states of mind. The distribution on the mental state space characterizes the decision maker's stochastic choice behavior.

More formally, let $\{\succcurlyeq_\omega \mid \omega \in \Omega\}$ be a set of preference relations on the set, H, of Anscombe and Aumann (1963) acts, and assume that they satisfy the axioms of expected utility theory. A menu, M, is a non-empty compact subset of Anscombe–Aumann acts. An act induced by M, denoted f_M, is an assignment to each $\omega \in \Omega$ of an act $h \in M$ such that $h \succcurlyeq_\omega h'$, for all $h' \in M$. We denote by F the set of acts induced by menus. Let $\hat{\succcurlyeq}$ be a preference relation on the set of all menus. Define the induced preference relation on F as follows: $f_M \succcurlyeq f_{M'}$ if $M \hat{\succcurlyeq} M'$. Broadly speaking, the main result of this paper is identifying necessary and sufficient conditions that yield the following representation: There exist a continuous, non-constant, real-valued function u on $\Omega \times H$ that is affine in its second argument and is unique up to positive linear transformation, and an essentially unique probability distribution η on Ω such that, for all $f_M, f_{M'} \in F$,

$$f_M \succcurlyeq f_{M'} \Leftrightarrow \sum_{\omega\in\Omega} \eta(\omega)\left[u(\omega, f_M(\omega)) - u(\omega, f_{M'}(\omega))\right] \geq 0.$$

Moreover, for every two acts h and h', the probability of choosing h over h' is given by

$$\Pr\left(h \mid \{h, h'\}\right) = \eta\left(\{\omega \in \Omega \mid u(\omega, h) > u(\omega, h')\}\right).$$

In the context of risk, this representation is similar to that of Dekel et al. (2001). However, the uniqueness of η is specific to our model.[2]

The theory developed in this paper is related not only to the literature on random choice but also to the literature on choice behavior when preference relations are incomplete, the literature on choice of menus, and the work on grades of indecisiveness.

Applying our model to menus of lotteries, we show that our theory implies the axioms of Gul and Pesendorfer (2006). Hence, the probability measure η generates their random utility and random choice model. All our preference relations are defined ex ante, at an earlier stage, before the actual choice among various acts/lotteries. In that stage, the decision maker chooses among menus of alternatives. We do not make explicit the later, ex post, choice, but, as indicated above, we assume that it is consistent with the expectations the decision maker has at the earlier stage. To make the connection between the two stages more explicit, one can follow Ahn and Sarver (2013), who join together the ex ante model of Dekel et al. (2001) with the ex post random choice of Gul and Pesendorfer (2006).

The representation of incomplete preferences under uncertainty specifies a set of probability–utility pairs and requires that one alternative be strictly preferred over another if and only if the former yields higher subjective expected utility than the latter according to each probability–utility in the set.[3] In this context, we identify states of mind with probability–utility pairs. When the alternatives are noncomparable, the choice may be random. Our model implies that the likelihood that one alternative is chosen over another is the measure (according to η) of the subset of the states of mind that prefer that alternative.

We also show that Minardi and Savochkin's (2015) notion of grades of indecisiveness between two Anscombe–Aumann acts, say f and g, can be represented by the probability η of the set $\{\omega \in \Omega \mid f \succ_\omega g\}$.

The model developed in this paper is related to the literature on probabilistic choice originated by Luce and Suppes (1965) and later developed by Loomes and Sugden (1995). Recently, Melkonyan and Safra (2016) axiomatized the utility components of two families of such preferences, where one family satisfies the independence axiom. Our paper complements and extends that model by characterizing the inherent probability distribution over the possible states of mind (possible tastes).

A more detailed discussion of the connections between this paper and these branches of the literature appears in Section 3, following the presentation of our theory in the next section. The proofs are relegated to Section 4.

2. Stochastic choice theory

2.1. The analytical framework: revealed preferences over mental acts induced by menus

2.1.1. Acts and preferences

Let X be a finite set of *outcomes*, and denote by $\Delta(X)$ the set of all probability measures on X. For each $p, q \in \Delta(X)$, and $\alpha \in [0, 1]$, define $\alpha p + (1-\alpha) q \in \Delta(X)$ by $(\alpha p + (1-\alpha) q)(x) = \alpha p(x) + (1-\alpha) q(x)$, for all $x \in X$.

Let S be a finite set of *material states* (or states of nature), and denote by H the set of all mappings from S to $\Delta(X)$. Elements of H are referred to as *acts*.[4] For all $h, h' \in H$, and $\alpha \in [0, 1]$, define $\alpha h + (1-\alpha) h' \in H$ by $\left(\alpha h + (1-\alpha) h'\right)(s) = \alpha h(s) + (1-\alpha) h'(s)$, for all $s \in S$, where the convex operation $\alpha h(s) + (1-\alpha) h'(s)$ is defined as above. Under this definition, H is a convex subset of the linear space $\mathbb{R}^{|X|\cdot|S|}$.

Let $\mathcal{P}$ be the set of all preference relations on H whose structure is depicted by the following axioms:

(A.1) (**Strict total order**) The preference relation $\succ$ is asymmetric and negatively transitive.

(A.2) (**Archimedean**) For all $h, h', h'' \in H$, if $h \succ h'$ and $h' \succ h''$, then $\beta h + (1-\beta) h'' \succ h'$ and $h' \succ \alpha h' (1-\alpha) h''$ for some $\alpha, \beta \in (0, 1)$.

(A.3) (**Independence**) For all $h, h', h'' \in H$ and $\alpha \in (0, 1]$, $h \succ h'$ if and only if $\alpha h + (1-\alpha) h'' \succ \alpha h' + (1-\alpha) h''$.

(A.4) (**Nontriviality**) $\succ$ is not empty.

By the expected utility theorem, a preference relation satisfies (A.1)–(A.4) if and only if there exists a nonconstant real-valued function, $w(x, s)$, on $X \times S$, unique up to cardinal unit-comparable transformation,[5] such that, for all $h, h' \in H$,[6]

$$h \succ h' \Leftrightarrow \sum_{s\in S} \sum_{x\in X} w(x, s)\left[h(s)(x) - h'(s)(x)\right] > 0.$$

[2] Sadowski (2013) obtained uniqueness of the probabilities in the model of Dekel et al. (2001) by enriching the model with objective states.

[3] See Galaabaatar and Karni (2013). In the case of incomplete preferences under risk, we identify states of mind with utility function and the analogous results are Dubra et al. (2004) and Shapley and Baucells (2008).

[4] See Anscombe and Aumann (1963).

[5] A function $\hat{w}(x, s)$ is said to be cardinal unit-comparable transformation of $w(x, s)$ if there exist a real number $b > 0$ and $a \in \mathbb{R}^S$ such that $\hat{w}(x, s) = bw(x, s) + a(s)$, for all $(x, s) \in X \times S$.

[6] See Kreps (1988).