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Rationalizing investors' choices

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ABSTRACT

Assuming that agents' preferences satisfy first-order stochastic dominance, we show how the Expected Utility paradigm can rationalize all optimal investment choices: the optimal investment strategy in any behavioral law-invariant (state-independent) setting corresponds to the optimum for an expected utility maximizer with an explicitly derived concave non-decreasing utility function. This result enables us to infer the utility and risk aversion of agents from their investment choice in a non-parametric way. We relate the property of decreasing absolute risk aversion (DARA) to distributional properties of the terminal wealth and of the financial market. Specifically, we show that DARA is equivalent to a demand for a terminal wealth that has more spread than the opposite of the log pricing kernel at the investment horizon.

1. Introduction

The von Neumann and Morgenstern Expected Utility Theory (EUT) has for decades been the dominant theory for making decisions under risk. Nonetheless, this framework has been criticized for not always being consistent with agents' observed behavior (e.g., the paradox of Allais, 1953; Starmer, 2000). In response to this criticism, numerous alternatives have been proposed, most notably dual theory (Yaari, 1987), rank-dependent utility theory (Quiggin, 1993) and cumulative prospect theory (Tversky and Kahneman, 1992). These competing theories differ significantly, but all three satisfy first-order stochastic dominance (FSD). Indeed, many economists consider violation of this property as grounds for refuting a particular model; see for example Birnbaum and Navarrette (1998), and Levy (2008) for empirical evidence of FSD violations. Recall also that although the original prospect theory of Kahneman and Tversky (1979) provides explanations for previously unexplained phenomena, it violates FSD. To overcome this potential drawback, Tversky and Kahneman (1992) developed cumulative prospect theory.

In the presence of a continuum of states, we show that the optimal portfolio in any behavioral theory that respects FSD can be rationalized by the expected utility setting, i.e., it is the optimal portfolio for an expected utility maximizer with an explicitly known concave utility function. This implied utility function is unique up to a linear transformation among concave functions and can thus be used for further analyses of preferences, such as to infer the risk aversion of investors.

A surprising feature is that we only assume that the preferences respect FSD, which contrasts with earlier results on the rationalization of investment choice under expected utility theory. Dybvig (1988a, Appendix A) and Peleg and Yaari (1975) among others, have worked on this problem assuming that preferences preserve second-order stochastic dominance (SSD). However, being SSD-preserving is quite a strong assumption, and while consistency with FSD is inherent, and even enforced, in most decision theories, this is not readily the case for SSD. For instance, rank dependent utility theory satisfies FSD but not SSD (Ryan, 2006), and the same holds true for cumulative prospect theory (see e.g., Baucells and Heukamp, 2006). Thus, our results show that for any agent behaving according to the cumulative prospect theory (i.e., for a "CPT investor") there is a corresponding expected utility maximizer with concave utility purchasing the same optimal portfolio, even if the CPT investor can exhibit risk seeking behavior with respect to losses. This approach, however, is not intended to dispense with alternative models to expected utility theory, as they have been developed mainly to compare gambles and not to deal with optimal portfolio selection per se.

Our results are rooted in the basic insight that under some assumptions, the marginal utility at a given consumption level is

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proportional to the ratio of risk-neutral probabilities and physical probabilities (Duffie, 2010). At first, it then seems obvious to infer a (concave) utility function and the risk aversion from the optimal consumption of the investor. However, the characterization that the marginal utility is proportional to the pricing kernel at a given consumption level is valid only if the utility is differentiable at this consumption level. This observation renders the rationalization of investment choices by the expected utility theory non-trivial, as there are many portfolios for which the implied utility is not differentiable at all consumption levels, such as the purchase of options or capital guarantee products. Furthermore, in a discrete setting (with a finite number of equiprobable states), the utility function that is consistent with optimal consumption is not unique. In this context, Peleg and Yaari (1975) give one potential implied utility, but there are many others. In the presence of a continuum of states, when the pricing kernel is continuously distributed, we are able to derive the unique (up to a linear transformation) concave utility function that is implied by the optimal consumption of any investor who respects FSD.

The proof of our main results builds on Dybvig's (1988a, 1988b) seminal work on portfolio selection. Instead of optimizing a value function, Dybvig (1988a) specifies a target distribution and solves for the strategy that generates the distribution at the lowest possible cost.[1] Here, we seek to infer preferences of consumers who are investing in the financial market. We show that if their portfolio satisfies some conditions, then it can be rationalized by expected utility theory with a non-decreasing and concave function. We assume that there is an infinite number of states in which it is possible to invest, whereas previous work considered a finite number of states. The assumption that we make is natural in the context of optimal portfolio selection problems. It allows us to obtain the uniqueness of the implied concave utility and to be able to use this inferred utility to estimate risk aversion. Inference of risk preferences from observed investment behavior has also been studied by Sharpe (2007) and Dybvig and Rogers (1997) for instance.[2] Sharpe (2007) assumes a static setting and relies on Dybvig's (1988b) results to estimate the coefficient of constant relative risk aversion for a CRRA utility based on target distributions of final wealth.

In this paper, we establish a link between expected utility theory (EUT) and all other theories that respects FSD. This connection can be used to estimate the agents' utility functions and risk aversion coefficients in a non-expected utility setting. Our approach in doing so is non-parametric and is based solely on knowledge of the distribution of optimal wealth and of the financial market. This is in contrast with traditional approaches to inferring utility and risk aversion, which specify an exogenous parametric utility function in isolation of the market in which the agent invests and then calibrate this utility function using laboratory experiments and econometric analysis of panel data.

It is widely accepted that the Arrow–Pratt measure of absolute risk aversion is decreasing with wealth. This feature – i.e., decreasing absolute risk aversion (DARA) – is often the motivation for using the CRRA utility instead of the exponential utility to model investors' preferences. In this paper, we show that the DARA property is completely characterized by a demand for final wealth W that exhibits more spread than a certain market variable (the opposite of the log pricing kernel). Our characterization of DARA can be used to empirically test DARA preferences based on observed investment decisions.

The paper is organized as follows. The introductory example in Section 2 explains in a simplified setting why a distribution of terminal wealth can always be obtained as the optimum of the maximization of expected utility for a risk-averse agent. The general setting is presented in Section 3 with the strong connection between law-invariance and first-order stochastic dominance. Section 4 shows that *any* distribution of terminal wealth can be obtained as the optimum of an expected utility maximizer with non-decreasing and (possibly non strictly) concave utility. Section 5 provides some applications of the results derived in Section 4. In particular, we illustrate how a non-decreasing concave utility function can be constructed to explain the demand for optimal investments in Yaari's (1987) setting. In Section 6, we show how to derive the coefficients of risk aversion directly from the choice of the distribution of final wealth and the financial market. In this section, we also explore the precise connections between decreasing absolute risk aversion and the variability of terminal wealth. In Section 7, we derive new utilities corresponding to well-known distributions and discuss their properties. Some proofs are relegated to Appendix.

2. Introductory example

Throughout this paper, we consider agents with law-invariant and non-decreasing preferences,[3] $V(\cdot)$. We say that $V(\cdot)$ is *non-decreasing* if, for consumptions X and Y satisfying $X \leqslant Y$, one has that $V(X) \leqslant V(Y)$. We say that $V(\cdot)$ is *law-invariant* if $X \sim Y$ implies that $V(X) = V(Y)$, where "$\sim$" reflects equality in distribution. This is often referred to as a "state-independent" set of preferences. We also assume that the agent's initial budget is finite.

In this section, we present an example in order to introduce the notation and to explain in a simplified setting (a space with a finite number of equiprobable states) why a distribution of terminal wealth can always be obtained as the optimum of the maximization of expected utility for a risk averse agent. We will also show the limitations of this discrete setting and how it fails to identify *the* implied concave utility function and implied risk aversion of the investor.

The introductory example takes place in a finite state space $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$ consisting of N equiprobable states (with probability $\frac{1}{N}$) at some terminal time T. Denote by $\frac{\xi(\omega_i)}{N}$ the initial (positive) cost at time 0 of the Arrow–Debreu security that pays one unit in the ith state, ω_i, at time T and zero otherwise. Let us call $\xi := (\xi_1, \xi_2, \dots, \xi_N)$ the pricing kernel where $\xi_i := \xi(\omega_i)$. It is clear that any state-contingent consumption $X := (x_1, x_2, \dots, x_N)$ (with $x_i := X(\omega_i)$) at time T writes as a linear combination of the N Arrow–Debreu securities.

The optimal investment problem of the agent with preferences $V(\cdot)$ is to find the optimal consumption X^* by solving the optimization problem,

$$\max_{X \mid E[\xi X]=X_0} V(X), \tag{1}$$

[1] It may indeed be more natural for an investor to describe her target distribution of terminal wealth instead of her utility function. For example, Goldstein et al. (2008) discuss how to estimate the distribution at retirement using a questionnaire. The pioneering work in portfolio selection by Markowitz (1952) is based solely on the mean and variance of returns and does not invoke utility functions. Black (1988) calls a utility function "a foreign concept for most individuals" and states that "instead of specifying her preferences among various gambles the individual can specify her consumption function".

[2] Under some conditions, Dybvig and Rogers (1997) infer utility from dynamic investment decisions. Our setting is static and well adapted to the investment practice by which consumers purchase a financial contract and do not trade thereafter.

[3] This assumption is present in most traditional decision theories including the von Neumann and Morgenstern expected utility theory, Yaari's dual theory (Yaari, 1987), the cumulative prospect theory (Tversky and Kahneman, 1992) and rank dependent utility theory (Quiggin, 1993).