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A dynamic extension of the Foster–Hart measure of riskiness

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ABSTRACT

We analyze the Foster–Hart measure of riskiness for general distributions in dynamic settings. The Foster–Hart measure avoids bankruptcy in the long run. It is not time-consistent.

1. Introduction

Foster and Hart (2009) introduce a notion of riskiness, or critical wealth level, for gambles with known distribution. Formally, the Foster–Hart measure of riskiness is given by the unique solution $R(X)$ of

$$E \log\left(1 + \frac{X}{R(X)}\right) = 0. \tag{1}$$

The Foster–Hart measure of riskiness $R(X)$ is defined for discrete random variables X on some probability space $(\Omega, \mathcal{F}, P)$ that satisfy $EX > 0$ and $P(X < 0) > 0$.

Riedel and Hellmann (2015) noticed that for general continuous distributions the defining equation does not necessarily admit a solution. In this case, the riskiness of sequences of discrete gambles that approximate the gamble with continuous distribution converges to the maximal loss of the gamble; Riedel and Hellmann (2015) thus suggest to use the maximal loss as the reasonable extension for the Foster–Hart measure when there exists no solution to Eq. (1).

In this paper, we study the extended Foster–Hart index of riskiness for general gambles in dynamic settings. As many financial applications require to quantify risk over time in a dynamic way, it seems natural and important to generalize the concept to a dynamic framework.

Dynamic measurement of risk plays an important role in the recent literature[1] since it allows, in contrast to the static case, to measure risk of financial positions over time. The arrival of new information can thus be taken into account. This is important for many situations; suppose, for instance, one faces a gamble that has its payments in, say, one month. In two weeks from now the information about this gamble might be much more precise which allows to adjust the risk assessment and to determine the risk more accurately. A static risk measure cannot do that. To cover such cases it is therefore crucial to be able to merge from static to dynamic risk measurement.

We thus set out to study the Foster–Hart measure of riskiness (or more precisely the extended Foster–Hart measure of riskiness defined in Riedel and Hellmann (2015)) in a dynamic framework. As a first step, we define the concept of conditional Foster–Hart riskiness for general probability spaces and filtrations. In the original work of Foster and Hart (2009) a somewhat dynamic approach is already needed to prove the no-bankruptcy result. Their approach, however, is rather intuitive than precise in a measure-theoretic sense. We provide here a more rigorous approach which allows us also to drop the assumption used in Foster and Hart (2009) that all gambles are multiples of a finite number of so-called basic gambles. Furthermore, we allow the extended Foster–Hart measure of riskiness to measure also gambles with potentially unbounded gains.

In the new framework, we show that Foster–Hart's no-bankruptcy result (and with it the operational interpretation) car-

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[1] See, among others, Detlefsen and Scandolo (2005) and Föllmer and Schied (2011), Chapter 11 for a detailed introduction to dynamic risk measures.

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ries over to general continuous distributions. The proof uses a different martingale argument which might be interesting in itself.

A desirable property of a dynamic risk measure is the notion of time-consistency. Time-consistency for dynamic risk measures is widely studied in the recent literature, see, among others, Riedel (2004), Roorda et al. (2005), Detlefsen and Scandolo (2005), Weber (2006) and Artzner et al. (2007). Roughly speaking a measure is time-consistent if it assigns a greater risk to one gamble than to another whenever it is known that the same holds true tomorrow. This property yields a consistent behavior of an agent who bases her decision on a time-consistent risk measure.

This property is not satisfied by many risk measures. In fact, the still most widely used Value at Risk has, besides many other undesirable properties, this inconsistence feature as it is shown in Cheridito and Stadje (2009). The same holds true for the dynamic Average Value at Risk. Cheridito and Stadje (2009), however, propose an alternative time-consistent version of the Value at Risk by composing one period Value at Risks over time.

On the other hand, a nice example for a time-consistent risk measure is given in Detlefsen and Scandolo (2005). They show that the dynamic entropic risk measure which is closely related to an agent with expected exponential utility preferences is time-consistent.

The dynamic version of the Foster–Hart measure of riskiness, however, does not satisfy the time-consistency condition. We show this by the use of a simple two period example. This example indicates a difference between the original static Foster–Hart measure and our dynamic version. In some instances the static Foster–Hart measure differentiates between two gambles, which are assigned to the same risk in every possible state of the world at a certain point in time by the conditional measure.

The paper is set up as follows: Section 2 introduces the dynamic framework as well as the dynamic extended Foster–Hart measure of riskiness. In Section 3 we give the more general no-bankruptcy result. Section 4 contains a counterexample which shows the time-inconsistency of the new defined measure. Finally, we prove the existence of the dynamic Foster–Hart index in Section 5.

2. The dynamic framework

In the following, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}}, P)$ be a filtered probability space. We denote by $\mathcal{A}_t$ the set of all $\mathcal{F}_t$-measurable random variables and consider a sequence of random variables (X_t) that is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$. In order to be able to measure the risk of X_t in every time period $s < t$, X_t has to satisfy all the conditions which define a gamble in Riedel and Hellmann (2015) given the filtration $(\mathcal{F}_s)$.

Definition 2.1. We call a random variable $X \in L^2$ on $(\Omega, \mathcal{F}, P)$ a gamble for the σ-field $\mathcal{F}_s \subset \mathcal{F}$ if X is bounded from below and satisfies $E[X|\mathcal{F}_s] > 0$ *a.s.* and $P(X_t < 0|\mathcal{F}_s) > 0$ *a.s.*

In the remainder, we assume that for $t > s$, X_t is a gamble for $\mathcal{F}_s$. We denote by $L_s(X_t)$ the maximal loss of X_t given the information at time s. Formally,

$$L_s(X_t) := \operatorname{ess\,inf}\{Z \in \mathcal{A}_s | P(-X_t > Z|\mathcal{F}_s) = 0 \quad a.s.\}.$$

We now embed the extended riskiness notion of Riedel and Hellmann (2015) in the dynamic framework. As time goes by, we learn something about the realization of the random variable and are therefore able to quantify the risk more precisely. Measuring the risk of X_t in every single time period $s < t$ yields a family of conditional risk measures $(\rho_s(X_t))_{s=1\ldots t-1}$, where every $\rho_s(X_t)$ is a $\mathcal{F}_s$-measurable random variable. For continuous random variables the equation

$$E\left[\log\left(1+\frac{X_t}{\rho_s(X_t)}\right)|\mathcal{F}_s\right] = 0 \tag{2}$$

does not always have a solution. Following the arguments of Riedel and Hellmann (2015),[2] this is the case on the set

$$\mathcal{B} := \left\{E\left[\log\left(1+\frac{X_t}{L_s(X_t)}\right)|\mathcal{F}_s\right] \geq 0\right\}.$$

As in the static case, on $\mathcal{B}$ the conditional maximal loss is the reasonable extension of the classical riskiness notion.

The next theorem shows that what we define later as the dynamic extended Foster–Hart riskiness is well defined.

Theorem 2.2. *There exists one and only one $\mathcal{F}_s$-measurable random variable $\rho_s(X_t) \geq L_s(X_t)$ that solves Eq. (2) on $\mathcal{B}^c$ and satisfies $\rho_s(X_t) = L_s(X_t)$ on $\mathcal{B}$.*

We give the technical proof of the theorem in Section 5.

We are now ready to give the definition of the dynamic extended Foster–Hart measure of riskiness.

Definition 2.3. The dynamic extended Foster–Hart measure of riskiness for a gamble X_t is the family of conditional risk measures $(\rho_s(X_t))_{s=1\ldots t-1}$, where each $\rho_s(X_t)$ is equal to the conditional maximal loss $L_s(X_t)$ on $\mathcal{B}$ and the solution to Eq. (2) on $\mathcal{B}^c$.

3. No-Bankruptcy result

The main result of Foster and Hart (2009) yields that a decision maker who rejects a gamble whenever his wealth is below the associated riskiness number avoids bankruptcy (with probability one). It is crucial not to lose this property (and with it the operational interpretation of the measure) when working with continuous distributed gambles.

We provide here the respective no-bankruptcy theorem for the extended Foster–Hart measure of riskiness.

Theorem 3.1. *Let (X_n) be a sequence of gambles that are uniformly bounded above by some integrable random variable $Y > 0$ and satisfy some minimal possible loss requirement, i.e. there exists $\epsilon > 0$ such that a.s.*

$$L_{n-1}(X_n) \geq \epsilon > 0$$

for all n. Let $W_0 > 0$ be the initial wealth and define recursively

$$W_{t+1} = W_t + X_{t+1}$$

if $E[\log(1 + X_{t+1}/W_t)|\mathcal{F}_t] \geq 0$ and

$$W_{t+1} = W_t$$

else. We then ensure no-bankruptcy, i.e.

$$P[\lim W_t = 0] = 0.$$

Proof. Throughout the proof we assume that all inequalities and equalities between random variables hold P-almost surely.

Note first that $W_t > 0$. This can be shown by induction. We have $W_0 > 0$. We have either $W_{t+1} = W_t$ which is positive by induction hypothesis, or $W_{t+1} = W_t + X_{t+1}$. In this case, the condition $E[\log(1 + X_{t+1}/W_t)|\mathcal{F}_t] \geq 0$ implies that

$$W_t \geq \rho_t(X_{t+1}) \geq L_t(X_{t+1}).$$

Thus, $W_t - L_t(X_{t+1}) \geq 0$. The maximal loss can only be obtained by the riskiness measure if the considered gamble is continuous. Therefore, if $\rho_t(X_{t+1}) = L_t(X_{t+1})$, we have $P(X_{t+1} = L_t(X_{t+1})|\mathcal{F}_t) = 0$. Hence, it holds that

$$W_{t+1} > W_t - L_t(X_{t+1}) \geq 0.$$

[2] For more details we refer to Section 5