Journal of Mathematical Economics 52 (2014) 50–56

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Multiple solutions in systems of functional differential equations☆

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ARTICLE INFO

Article history:
Received 1 February 2013
Received in revised form
19 March 2014
Accepted 22 March 2014
Available online 3 April 2014

Keywords:
Delay differential equations
Advance differential equations
Existence
Indeterminacy

ABSTRACT

This paper proposes conditions for the existence and uniqueness of solutions to systems of linear differential or algebraic equations with delays or advances, in which some variables may be non-predetermined. These conditions represent the counterpart to the Blanchard and Kahn conditions for the functional equations under consideration. To illustrate the mathematical results, applications to an overlapping generations model and a time-to-build model are developed.

1. Introduction

Two common characteristics of many dynamic models in economics are that the initial values of some variables are unknown and that certain asymptotic properties – notably convergence toward a steady state – must be accounted for. Mathematically, these are boundary value problems. The analytical resolution method consists of projecting the trajectory onto the stable eigenspace of the dynamic system. By comparing the dimensions of the space of non-predetermined variables with those of the unstable eigenspace, one can deduce the properties of the existence and determinacy of a solution to the system under consideration (Blanchard and Kahn, 1980; Buiter, 1984). The equilibrium is said to be indeterminate when there is more than one solution, potentially causing sunspot fluctuations to appear (Azariadis, 1981; Benhabib and Farmer, 1999). However, the mathematical theorems that characterize these properties were only established for systems of finite dimensions comprising ordinary differential equations (ODEs) or difference equations. In this paper, we generalize these theorems to include some systems of delay or advanced differential equations (DDEs or ADEs).

As Burger (1956) pointed out, many dynamic systems in economics can be written as DDEs. Since his work, DDEs have been used in the demographic economics, vintage capital, time-to-build, and monetary policy literatures (see Boucekkine et al., 2004 for an excellent survey of the use of DDEs in economics). However, for want of a theorem, researchers have had to either confine their work to very specific cases where the stability properties of the dynamics can be proven[1] or use numerical methods or other mathematical tools (most notably optimal control with the Hamilton–Jacobi–Bellman equation).[2]

DDE systems, which are characterized by a stable manifold of infinite dimensions, have generated an abundance of mathematical literature (see the textbooks by Bellman and Cooke, 1963, Diekmann et al., 1995). However, the existing theorems are only valid for systems where all the variables are predetermined and defined as continuous functions. Our first objective is to extend these theorems to cases where some variables are non-predetermined (i.e., their past values are given but their value when the system is initiated is unknown) and to cases where some predetermined

☆ We thank M. Bambi, J.-P. Drugeon, F. Kubler, and an anonymous reviewer for their stimulating suggestions and comments. H. d'Albis acknowledges financial support from the European Research Council (ERC Stg Grant DU 283953). H.J. Hupkes acknowledges support from the Netherlands Organization for Scientific Research (NWO) (639.031.139). The usual disclaimer applies.

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[1] See, among others, Gray and Turnovsky (1979), Boucekkine et al. (2005), Bambi (2008), Augeraud-Véron and Bambi (2011) and d'Albis et al. (2012b).

[2] See Fabbri and Gozzi (2008), Freni et al. (2008), Boucekkine et al. (2010), Federico et al. (2010) and Bambi et al. (2012a).

http://dx.doi.org/10.1016/j.jmateco.2014.03.010
0304-4068/

variables are discontinuous. To do so, we use the mathematical results of d'Albis et al. (2012a). In that paper, we defined an operator that acts on a multivalued space and studied its properties. In the present paper, we use the properties of this operator to rewrite a spectral projection formula according to the initial conditions and compute the jump made by non-predetermined variables. We set the projection on the unstable manifold to zero and deduce the magnitude of the jump that nullifies the projection on the unstable manifold. The spectral projection formula then enables us to establish the conditions for the existence and uniqueness of a solution. Most notably, we prove that it is possible to come to a conclusion by comparing the dimensions of the space of the unknown initial conditions to those of the unstable eigenspace. Our results also apply to systems of algebraic equations with delays if their nth derivative is a DDE. In this case, the constraints imposed by such equations must be accounted for by the conditions for existence and uniqueness.

Our second objective is to extend these theorems to differential equations with advances. Systems of ADEs are more similar to ODE systems as they have a stable eigenspace of finite dimensions. We demonstrate that the solution is generated by a finite number of eigenvalues simply by projecting the trajectory onto the stable eigenspace. Conditions for existence and determinacy are obtained by comparing the number of roots with negative real parts to the number of initial conditions. We further study the case of systems that include algebraic equations and define the additional constraints that must be considered.

In Section 2, we present the kind of equations we are interested in and relate them to the literature in economics. In Section 3, our main theorems are presented. The conditions for the existence and uniqueness of solutions to systems of DDEs are in Section 3.1, whereas those of systems of ADEs are in Section 3.2. In Section 4, we solve two economic models in order to illustrate our results and show how to apply our theorems. An overlapping generations model whose dynamics are given by an algebraic equation with delay is studied in Section 4.1 and the decentralized economy of a time-to-build model that can be written with a system of DDEs is studied in Section 4.2. Section 5 concludes.

2. Presentation of the problem

To fix matters, we consider a DDE. Letting $t \in \mathbb{R}_+$ denote time, the dynamic problem can be written as:

$$\begin{cases} x'(t) = \int_{t-1}^{t} d\mu\,(u-t)\,x(u), \\ x(\theta) = \bar{x}(\theta) \quad \text{given for } \theta \in [-1, 0], \end{cases} \tag{1}$$

where x is a variable with initial value given by a continuous function over the interval $[-1, 0]$, x' denotes its derivative with respect to time, and μ is a measure on $[-1, 0]$. Eq. (1) features dynamics that depend on past variables (i.e., delays) on the interval $[t-1, t]$.[3] In economics, the Johansen (1959) and Solow (1960) vintage capital models are well known examples of dynamic problems described by (1). Classical results for such dynamics are presented in Diekmann et al. (1995).

In economic models, we may have other types of systems. Herein, we will consider three dynamics that differ from (1). First, we study algebraic equations with delay that reduce to DDEs upon (a finite number of) differentiations with respect to time. This problem can be written as:

$$\begin{cases} x(t) = \int_{t-1}^{t} d\mu\,(u-t)\,x(u), \\ x(\theta) = \bar{x}(\theta) \quad \text{given for } \theta \in [-1, 0]. \end{cases} \tag{2}$$

The main difference with the DDE presented above comes from a discontinuity that is allowed at time $t = 0$: $x\left(0^+\right)$ is given but may be different from $x\left(0^-\right)$. Indeed, $x\left(0^+\right)$ is given through the algebraic equation:

$$x(0^+) = \int_{-1}^{0} d\mu\,(u)\,x\,(u)\,. \tag{3}$$

To summarize, the initial value is provided by $x\left(0^+\right)$ and a continuous function over the interval $[-1, 0)$ where $x\left(0^-\right)$ exists. In both problems (1) and (2), the variable is predetermined and is usually backward-looking. Examples of such dynamics are given in Benhabib (2004) and d'Albis et al. (2014) for interest rate policy models and by de la Croix and Licandro (1999) and Boucekkine et al. (2002) for vintage human capital issues. We will study the latter as an illustrative example in Section 4.1.

The second kind of dynamics we consider allows for non-predetermined variables (i.e., forward-looking variables) that do not have a given initial value at time $t = 0$. For a DDE, this dynamic problem can be written as:

$$\begin{cases} x'(t) = \int_{t-1}^{t} d\mu\,(u-t)\,x\,(u)\,, \\ x(\theta) = \bar{x}(\theta) \quad \text{given for } \theta \in [-1, 0). \end{cases} \tag{4}$$

The initial value is now given by a function that is continuous on $[-1, 0)$ and bounded in 0. Growth theory examples of such dynamics can be found in d'Albis et al. (2012b) and Bambi et al. (2012a,b). An example for vintage capital theory can be found in Jovanovic and Yatsenko (2012).

Finally, the third type of dynamics considers equations with advances rather than delays. For instance, an ADE can be written as:

$$x'(t) = \int_{t}^{t+1} d\mu\,(u-t)\,x\,(u)\,. \tag{5}$$

ADEs appear as the Euler equation of some vintage capital models studied using optimal control (Boucekkine et al., 2005) or dynamic programming (Boucekkine et al., 2010). Depending on whether or not $x(0)$ is given, the dynamics characterize a backward-looking or a forward-looking variable. Finally, algebraic equations with advances can also be considered in monetary theory models, as in d'Albis et al. (2014).

3. Main theorems

In this section, we study functional differential–algebraic systems with delays and then we study those with advances.

3.1. Functional systems with delays

Let us consider the following linear system:

$$\begin{cases} \mathbf{x}_0'(t) = \int_{t-1}^{t} d\bar{\mu}_1\,(u-t)\,W(u), \\ \mathbf{x}_1(t) = \int_{t-1}^{t} d\bar{\mu}_2\,(u-t)\,W(u), \\ \mathbf{y}'(t) = \int_{t-1}^{t} d\bar{\mu}_3\,(u-t)\,W(u), \\ \mathbf{x}_i(\theta) = \bar{\mathbf{x}}_i(\theta) \quad \text{given for } \theta \in [-1, 0] \text{ and } i = \{0, 1\}, \\ \mathbf{y}(\theta) = \bar{\mathbf{y}}(\theta) \quad \text{given for } \theta \in [-1, 0). \end{cases} \tag{6}$$

[3] Note that the largest delay is normalized to one even though it could be any positive real number. However, we do not consider systems with infinite delays as their characteristic roots may not be isolated.