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A graph theoretic approach to markets for indivisible goods

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ABSTRACT

Many important markets, such as the labor market and the housing market, involve goods that are both indivisible and of budgetary significance. We introduce new graph theoretic objects ideally suited to analyzing such markets. We show that the minimum equilibrium price is characterized by a certain optimization problem on these graph theoretic objects.

1. Introduction

Jobs, houses and other large durable goods are generally indivisible, heterogeneous, and of budgetary significance. Each of these characteristics poses a challenge for model builders. Indivisibility imposes integer constraints. Heterogeneity leads to a non-trivial problem of matching buyers and sellers. Expense makes it difficult to justify a local linear approximation to utility.

Given these difficulties progress in understanding markets of this kind has proceeded slowly. Kaneko (1982) was first to establish conditions for existence of equilibria.[1] Demange and Gale (1985) showed under much the same conditions that the set of equilibrium prices is a lattice with maximal and minimal elements. Demange and Gale also show that the minimum price equilibrium cannot be manipulated by buyers, as well as some basic comparative static properties of the minimum price equilibrium (e.g. minimum equilibrium prices rise when more buyers are introduced).[2] Several authors have devised algorithms for computing equilibria in such models.[3]

In this paper, we provide a simple characterization of the minimum price competitive equilibrium in an allocation market with non-transferable utility. It is well known (Demange and Gale, 1985), that the minimum price competitive equilibrium is supported by a set of indifference conditions. Any good whose price is above reservation must be demanded by some buyer assigned to some other good. Otherwise one could lower prices. We use this insight to motivate the study of graph-allocation pairs, which we call GA-structures, where each GA-structure combines an allocation of goods to buyers and a directed graph that indicates which goods to use to price other goods. Together these two objects generate price vectors: the graph determines which goods are connected by indifference and the allocation determines whose indifference should be used in computing prices.

Our main result is that the minimum competitive equilibrium price may be derived by first maximizing prices over graphs for a given allocation and them minimizing over allocations. These operations in tandem ensure that buyers are allocated to their preferred goods. Intuitively, if at a given set of prices a buyer prefers another good to the one she is allocated, then one can either incorporate that indifference condition into the price structure, which would raise prices, or one could reallocate the buyer to her preferred good, which would tend to reduce that buyer's willingness to pay for other goods and thereby lower prices.

Indifference relations have played a major role in models of heterogeneous goods at least since the rent-gradient model of Ricardo (1817). Indifference relations appear in the analysis of Demange and Gale (1985), and are central to the approaches of Alkan (1989) and Kaneko et al. (2006) among many others.[4] What is different in the present case is the separation of the indifference relationship into two parts: a graph and an allocation. The graph focuses

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1 Quinzii (1984), Gale (1984), and Kaneko and Yamamoto (1986), and Alkan (1989) also provide existence proofs. Crawford and Knoer (1981) sketch a proof of existence for a version of their model with non-transferable utility.

2 See also Miyake (1994).

3 The ascending auction mechanisms of Crawford and Knoer (1981) and Demange et al. (1986) solve for the minimum price equilibrium in a discretized version of the model. Alkan (1989) constructs an algorithm under the assumption of piecewise linear utility. Miyake (2003) presents an algorithm that converges to the exact solution of the general model.

4 See Mishra and Talman (2010) for a fairly general analysis of indifference relationships in the case of transferable utility.

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only on the goods and represents what goods are related by indifference. The allocation determines who is indifferent. This distinction between the who and the what allows us to operate on these two aspects of the indifference relationship separately, to minimize with respect to who is indifferent and to maximize with respect to which goods are related by indifference.

Experience with the fundamental theorem of welfare economics shows how useful such a link between equilibrium theory and optimization theory can be. Optimization problems are simpler and better understood. They also do not require one to explicitly consider demand, supply, or the balance between them. In this paper, we use the link between optimization theory and equilibrium theory to characterize local comparative statics. We provide a chain rule for such local comparative statics. This chain rule establishes that small discrete shocks have local effects in the model. In contrast, with divisible goods, infinitesimal shocks have global effects: even the smallest change to the supply or demand for one good tends to affect the price of every other good in the economy.

The remainder of the paper is structured as follows. Section 2 discusses some related literature. Section 3 presents the basic model. Section 4 presents an example that illustrates the main objects of our analysis. Section 5 introduces GA-structures. Section 6 characterizes the minimum equilibrium price as the solution to a max–min problem on these structures. Section 7 characterizes the minimum price equilibrium allocation in a similar manner. Section 8 uses these characterizations to study the local dependence of minimum price equilibria on the economic environment. Section 9 reverses the roles of buyers and sellers to characterize the maximum price competitive equilibrium, and then shows how to manipulate sellers' reservation values to recover the complete set of equilibria. Section 10 concludes.

2. Related literature

The most common approaches to studying indivisibilities in markets for heterogeneous goods is either to assume linear utility or to make assumptions that smooth away the discreteness. Neither approach is satisfactory for the study of labor markets, housing markets or the markets for other large durable goods.

An example of the first approach is the model of Shapley and Shubik (1972). They showed that with linear utility the competitive equilibrium allocation in a market for heterogeneous, indivisible goods is equivalent to the problem of a social planner allocating goods so as to maximize the sum of utilities. This social planner's problem takes the form of the linear programming problem studied by Koopmans and Beckmann (1957).

The assumption of linear utility and the resulting absence of wealth effects, however, is problematic in many applications, especially if the good in question is an expensive one such as a house. In the linear case, the social planner allocates goods based on some fixed notion of how much each agent desires each good. If a poor agent values a sea-view more than a rich agent, the planner will allocate a mansion by the sea to the poor agent. We do not, however, see many poor agents living in sea-side mansions. What is missing is the effect of diminishing marginal utility of wealth that leads the rich to be willing to pay more than the poor for the nicest homes. To include these effects it is necessary to consider utility functions that are non-linear in wealth.

An example of smoothing is Rosen's (1974) hedonic pricing model. It also prices heterogeneous goods given heterogeneous buyers and sellers. While goods themselves are indivisible, Rosen makes the assumption that there is a continuous density over characteristic bundles and that within this space one can adjust each characteristic while fixing the others. This assumption smooths the type space allowing the use of the tools of calculus. In many applications, however, the type space may not be dense enough to allow such adjustments. In housing markets, for example, location is one of the most important characteristics. It is not generally possible to adjust location while keeping all other characteristics fixed, nor is it generally possible to alter characteristics of homes while maintaining a fixed location without incurring substantial costs. Moreover, there is ample evidence in the urban economics literature that hedonic prices vary with location.[5]

All of this makes the study of allocation markets with wealth effects important. We discussed some of the prior work in the introduction. Allocation problems arise naturally in a number of areas in economics. In the housing literature, the minimum equilibrium price vector is similar to the rent gradient found in Ricardo (1817), Alonso (1964), and Roback (1982). Models in this tradition tend to limit the heterogeneity in buyers or houses in order to keep the model tractable. At the same time, however, this simplicity allows them to go further than we will in modeling the supply side of the market. In the auction and mechanism design literature, our equilibrium is similar to a second price auction or a Vickrey–Groves–Clarke mechanism. These models almost always assume transferable utility. An exception is the paper by Demange and Gale (1985) cited above.

3. The model

We work with a variant of the model in Demange and Gale (1985). Demange and Gale simplify the exposition and the analysis of allocation markets by removing all reference to budget constraints. This assumption eliminates the need to discuss what transactions are feasible for each agent at each set of prices and also ensures that the choice correspondences are continuous.[6]

There is a set of buyers $x_a \in X$, $1 \leq a \leq m$, and a set of indivisible goods $y_i \in Y$, $1 \leq i \leq n$. The goods are initially held by the sellers. Buyers may purchase the indivisible goods from sellers by making a transfer in terms of a homogeneous, perfectly divisible, numeraire good, which may be thought of as money. Sellers choose only whether or not to sell. They do not purchase the indivisible goods from other sellers. We assume that $n \geq m$ so that it is possible to match each buyer with a good.[7]

We assume that buyers can derive utility from at most one element of Y. The payoff for buyer x_a depends on the good that buyer purchases and the size of the transfer that the buyer makes to the seller. This payoff is summarized by the utility function $U_a : Y \times \mathbb{R} \to \mathbb{R}$, where $U_a(y_i, p_i)$ is the utility to x_a from the purchase of y_i at the price p_i.

Let $p \in \mathbb{R}^n$ denote the vector of goods prices. Each seller wishes to obtain the highest possible price above a reservation level. Let $r \in \mathbb{R}^n$ denote the vector of seller reservation prices. The supply side is trivial: each seller prefers to hold on to their good for any $p_i < r_i$ and to sell for any $p_i > r_i$. The seller is indifferent when $r_i = p_i$.[8] Choosing $r \geq 0$ will ensure that all prices are positive if so desired.

Given any price vector $p \in \mathbb{R}^n$, the demand correspondence $D_a(p)$ specifies members of Y that maximize the utility of x_a:

$$D_a(p) = \{y_i \in Y | U_a(y_i, p_i) \geq U_a(y_k, p_k) \text{ for all } y_k \in Y\}.$$

5 See, for example, Meese and Wallace (1994).

6 With budget constraints, consumers' choice correspondences may not be continuous, and therefore the demand correspondence may fail to be upper-hemicontinuous. Assumptions (such as the Inada conditions) may need to be made to ensure that the constraints are not binding in equilibrium, but these do not add insight.

7 This is without loss of generality. The possibility that a buyer may choose not to make a purchase can be captured by associating a subset of goods with exit.

8 Since we will be interested in minimum price competitive equilibria, the exact form of a seller's utility does not matter so long as it is increasing in the transfer and there is a point r_i at which seller i is indifferent between selling and holding.