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Single-basined choice☆

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ABSTRACT

Single-basined preferences generalize single-dipped preferences by allowing for multiple worst elements. Single-dipped and single-basined preferences have played an important role in areas such as voting, strategy-proofness and matching problems. We examine the notion of single-basinedness in a choice-theoretic setting, with the set of all compact convex subsets of $\mathbb{R}^n$ as the domain of choice sets. In conjunction with independence of irrelevant alternatives, single-basined choice implies a structure that conforms to the motivation underlying our definition. We establish the consequences of requiring single-basined choice correspondences to be upper semicontinuous. Moreover, we extend our results to larger domains of non-convex sets.

1. Introduction

Single-peakedness has a long history in economic theory. It goes back as far as Black (1948) who shows that if preferences are restricted to those that are single-peaked, then the majority rule generates transitive social preferences. Other important contributions include Inada (1969) and Sen (1970) who provide related value restrictions that focus on single-peaked preference profiles. See also Moulin (1980) and Sprumont (1991) for applications in the context of strategy-proofness. There is no need to restrict the notion of single-peakedness to a single dimension. Generalizations to higher dimensions are employed, for instance, by Barberà et al. (1993), Barberà and Jackson (1994), Dutta et al. (2002), Ehlers and Storcken (2008) and Le Breton and Weymark (2011). Ballester and Haeringer (2011) characterize one-dimensional single-peaked preference profiles by providing necessary and sufficient conditions for the existence of a single ranking such that all preferences in the profile are single-peaked with respect to this ranking.

Single-peakedness and its generalization single-plateauedness are analyzed in a choice-theoretic setting by Bossert and Peters (2009, 2013). The notion of single-peakedness is too restrictive if choices are permitted to be multi-valued. This generalization is applied by Moulin (1984), Berga (1998), Ehlers (2002a), Barberà (2007) and Berga and Moreno (2009), among others.

The natural counterpart of single-peakedness is single-dippedness, where preferences are such that there is a single 'dip' rather than a single peak, and we refer to its generalization to environments that permit multiple dips as single-basinedness. Single-dipped preferences frequently appear when a public bad is to be located and individuals are assumed to prefer a larger distance from the bad to being closer to its chosen location. If the preferences are interpreted as those of the representatives of a community or neighborhood, multiple dips appear to be plausible because, in this case, it is desirable to keep the bad at a distance not only from a specific location but also from an entire region. The relevant literature includes Kunreuther and Kleindorfer (1986), Klaus et al. (1997), Peremans and Storcken (1999), Klaus (2001), Ehlers (2002b), Lescop (2007), Besfamille and Lozachmeur (2010), Barberà et al. (2012), Öztürk et al. (2013, forthcoming) and Manjunath (forthcoming).

The purpose of this paper is to provide a choice-theoretic basis for single-basinedness (and, as a special case, single-dippedness). So far, neither single-dipped nor single-basined choice has been thoroughly analyzed. Thus, our contribution goes beyond a mere extension of existing results on single-dipped choice to single-basined choice: the examination of single-dippedness in a choice-theoretic setting alone is, in itself, novel. We work within a

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Euclidean space that can be of any (fixed) dimension and study choice correspondences that select a non-empty and compact subset of chosen elements from each non-empty, compact and convex subset of $\mathbb{R}^n$, the Euclidean n-dimensional space. As is common in this type of literature, we concentrate on choice correspondences that satisfy independence of irrelevant alternatives, a contraction consistency condition that is necessary (but, in general, not sufficient) for the rationalizability of a choice correspondence by an ordering; see, for instance, Richter (1966, 1971). In view of the typical applications of single-dippedness and single-basinedness, this focus is suitable. Choice rules that violate this fundamental independence condition are, in our opinion, not very appealing and, thus, we eliminate them from consideration not because independence of irrelevant alternatives is a necessary condition for rational choice but because including non-independent choice functions would allow for choice procedures that are rather poorly behaved.

We additionally impose the condition of single-basinedness on a choice correspondence: this condition says that if x is revealed preferred to y, then also all points on the straight line through x and y, further away from y than x, are revealed preferred to y. We then show that such a choice correspondence either has a basin, which is a (convex) set of worst points, or it does not have a basin, in which case it always picks boundary points from a choice set. Also, if the choice correspondence is upper semicontinuous and has a basin, then this basin is closed. Moreover, we show that adding upper semicontinuity results in the choice correspondence assigning the maximizers of a quasi-convex function.

In the final section of the paper we introduce a modification of the single-basinedness condition that allows the extension of our results to larger domains of compact (but not necessarily convex) choice sets under the weak axiom of revealed preference.

2. Independent choice correspondences

Suppose $n \in \mathbb{N}$ is fixed and define $\mathcal{C} = \{C \subseteq \mathbb{R}^n \mid C \text{ is non-empty, compact and convex}\}$. A choice correspondence is a mapping $\varphi: \mathcal{C} \twoheadrightarrow \mathbb{R}^n$ such that $\emptyset \neq \varphi(C) \subseteq C$ and $\varphi(C)$ is compact for all $C \in \mathcal{C}$. The choice of the domain $\mathcal{C}$ is motivated by both economic and technical considerations. Compactness is a standard requirement, especially when considering choice rationalized by a preference relation or utility function. Convexity ensures that mixtures (or lotteries) of outcomes can be accommodated. The convexity assumption can be relaxed; see Section 8.

The direct revealed preference relation R_φ of φ is defined as follows. For all $x, y \in \mathbb{R}^n$,

$$xR_\varphi y \Leftrightarrow \text{there exists } C \in \mathcal{C} \quad \text{such that } x \in \varphi(C) \text{ and } y \in C.$$

The asymmetric part of R_φ is denoted by P_φ, and I_φ is the symmetric part of R_φ.

A generalized version of Samuelson's (1938) weak axiom of revealed preference can be stated as follows.

Weak axiom of revealed preference. For all $C \in \mathcal{C}$ and for all $x, y \in C$, if $xR_\varphi y$ and $y \in \varphi(C)$, then $x \in \varphi(C)$.

There are numerous equivalent definitions of the weak axiom of revealed preference that can be found in the literature; see, for instance, Bossert and Suzumura (2010, p. 17). One of these is the equality of P_φ and the so-called direct revealed strict preference relation R^*_φ, defined as follows. For all $x, y \in \mathbb{R}^n$,

$$xR^*_\varphi y \Leftrightarrow \text{there exists } C \in \mathcal{C}$$
$$\text{such that } x \in \varphi(C) \text{ and } y \in C \setminus \varphi(C).$$

See Bossert and Peters (2013) for further details.

In our framework, the weak axiom of revealed preference is equivalent to independence of irrelevant alternatives, which is a contraction-consistency condition imposed on a choice correspondence. It is often referred to as Arrow's choice axiom (see Arrow, 1959) but, as Shubik (1982, pp. 420–421 and p. 423, footnote 2) remarks, the axiom already appears in 1950 in an informal note authored by Nash. A version for single-valued choice is due to Nash (1950) in the context of axiomatic bargaining theory.

Independence of irrelevant alternatives. For all $C, D \in \mathcal{C}$, if $D \subseteq C$ and $D \cap \varphi(C) \neq \emptyset$, then $\varphi(D) = D \cap \varphi(C)$.

For future reference, we note that, because our domain $\mathcal{C}$ is closed under intersection (that is, for all $C, D \in \mathcal{C}$, the intersection $C \cap D$ is also in $\mathcal{C}$ whenever this intersection is non-empty), the weak axiom of revealed preference is equivalent to independence of irrelevant alternatives; see Hansson (1968) for a generalization of this observation. We state this known result without proving it here; an explicit proof is provided in Bossert and Peters (2013).

Lemma 1. *A choice correspondence $\varphi: \mathcal{C} \twoheadrightarrow \mathbb{R}^n$ satisfies independence of irrelevant alternatives if and only if φ satisfies the weak axiom of revealed preference.*

As is well-known, the weak axiom of revealed preference implies independence of irrelevant alternatives even if the domain of a choice correspondence is not closed under intersection. As a consequence of Lemma 1, we can use independence of irrelevant alternatives and the weak axiom of revealed preference interchangeably.

3. Single-basined choice correspondences

For distinct $x, y \in \mathbb{R}^n$, $[x, y, \rightarrow)$ is the half-line through y starting at x and $[x, y]$ is the line segment with end points x and y. The (relatively) half-open sets $[x, y)$ and $(x, y]$, and the (relatively) open set (x, y) are defined analogously in the usual way. The boundary of $C \in \mathcal{C}$ is denoted by $bd(C)$ and the interior of C is $\text{int}(C)$. The convex hull of C is $conv(C)$ and the closure of a subset D of $\mathbb{R}^n$ is denoted by $cl(D)$. The convergence of a sequence of sets in $\mathcal{C}$ is defined in terms of the Hausdorff metric for compact subsets of $\mathbb{R}^n$. The function $d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ denotes the Euclidean distance in $\mathbb{R}^n$.

Single-basinedness of a choice correspondence is defined as follows.

Single-basinedness. For all distinct $x, y \in \mathbb{R}^n$, if $xR_\varphi y$, then $zR_\varphi y$ for all $z \in [y, x, \rightarrow)$ with $z \notin [y, x)$.

Thus, single-basinedness demands that if a point x is directly revealed preferred to another point y, then any point z that is located on the half-line starting at y and passing through x and that is, moreover, at least as far away from y as x, is directly revealed preferred to y.

Among other things, the next lemmas ensure that single-basinedness together with independence of irrelevant alternatives implies that the direct revealed preference relation is reflexive, complete and transitive; hence, that it is an ordering.

Lemma 2. *Let the choice correspondence $\varphi: \mathcal{C} \twoheadrightarrow \mathbb{R}^n$ satisfy independence of irrelevant alternatives and single-basinedness, and let $C \in \mathcal{C}$. If $\varphi(C) \cap \text{int}(C) \neq \emptyset$, then $\varphi(C) = C$.*

Proof. We need to prove that $C \subseteq \varphi(C)$. Let $x \in \varphi(C) \cap \text{int}(C)$. For any $y \in C \setminus \{x\}$, there exists $z \in [y, x, \rightarrow) \cap C$ such that $x \in (y, z)$ because $x \in \text{int}(C)$. We thus have $xR_\varphi z$ because $x \in \varphi(C)$ and $z \in C$. By single-basinedness, it follows that $yR_\varphi z$. Thus, there exists $C' \in \mathcal{C}$ such that $y \in \varphi(C')$ and $z \in C'$. By independence of irrelevant alternatives, $y \in \varphi([y, z])$ and hence $yR_\varphi x$ because $x \in [y, z]$. Because $y \in C$ and $x \in \varphi(C)$, the weak axiom of revealed preference implies $y \in \varphi(C)$. ■

Lemma 3. *Let the choice correspondence $\varphi: \mathcal{C} \twoheadrightarrow \mathbb{R}^n$ satisfy independence of irrelevant alternatives and single-basinedness, let $C \in \mathcal{C}$ and let $x, y, z \in C$ be such that $z \in (x, y)$. If $z \in \varphi(C)$, then $[x, y] \subseteq \varphi(C)$.*