Mathematical Social Sciences 95 (2018) 1–8

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/mss

Decomposing bivariate dominance for social welfare comparisons[☆]

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HIGHLIGHTS

- We study how bivariate dominance can be decomposed into two elementary operations.
- Suitable diminishing transfers and correlation-increasing switches are explicitly described.
- Our constructive algorithm determines how much mass has to be moved by diminishing transfers.

ARTICLE INFO

Article history:
Received 13 December 2017
Received in revised form 8 June 2018
Accepted 28 June 2018
Available online 5 July 2018

ABSTRACT

The principal dominance concept for inequality-averse multidimensional social welfare comparisons, commonly known as lower orthant dominance, entails less or equal mass on all lower hyperrectangles of outcomes. Recently, it was shown that bivariate lower orthant dominance can be characterized in terms of two elementary mass transfer operations: diminishing mass transfer (reducing welfare) and correlation-increasing switches (increasing inequality). In this paper we provide a constructive algorithm, which decomposes the mass transfers into such welfare reductions and inequality increases.

1. Introduction

Dominance concepts are increasingly used for multidimensional comparisons of social welfare, inequality, and poverty (see, e.g., Aaberge and Brandolini, 2014).[1] Such concepts are appealing, since they provide comparisons of the overall attainment of groups, which are robust for broad classes of individual and social preferences over the (multidimensional) outcomes.

An important and frequently used dominance concept for inequality-averse multidimensional social welfare comparisons is *lower orthant dominance*. The idea of using orthant dominance – and related (less restrictive) concepts – for inequality-averse social welfare comparisons was popularized by Atkinson and Bourguignon (1982), and it has been significantly developed and refined in several articles (see, e.g., Bourguignon and Chakravarty, 2003, Duclos et al., 2006, 2007, Gravel et al., 2009, Gravel and Mukhopadhyay, 2010, and Muller and Trannoy, 2011).[2]

Suppose that there are multiple dimensions of welfare and that, for each dimension, a wellbeing indicator can take a finite number of possible values.[3] We can then describe a population distribution

☆ The authors are grateful to an associate editor and two anonymous referees of this journal for their remarks that helped to improve the paper. Support from Independent Research Fund Denmark (Grant-IDs: DFF–1327-00097 and DFF–6109-000132) is acknowledged, and the third author acknowledges support from the Spanish Ministerio de Economía y Competitividad under Project ECO2015-66803-P (MINECO/FEDER).

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1 Stochastic dominance is not only useful in welfare economics, but also in many other fields. It is, for example, an important tool in decision theory (see, e.g., Levy, 1992 or Müller and Stoyan, 2002), finance (see, e.g., Sriboonchita et al., 2009), as well as in probability theory and statistics (see, e.g., Silvapulle and Sen, 2011).

2 Note that lower orthant dominance has sometimes been referred to as "first order dominance", particularly in the welfare economics literature. In order not to risk confusion with the usual stochastic order – the natural dominance concept for multidimensional social welfare comparisons with ordinal data (see, e.g., Arndt et al., 2012, Østerdal, 2010, and Range and Østerdal, 2017) – we use the term lower orthant dominance as customary in the probability theory literature (e.g. Shaked and Shanthikumar, 2007).

3 Chakravarty and Zoli (2012) mention a number of applications in which a wellbeing indicator is discrete by nature.

https://doi.org/10.1016/j.mathsocsci.2018.06.005
0165-4896/

by a probability mass function over the outcomes; i.e., by a function that assigns to each outcome the probability that a randomly selected individual obtains that outcome (or, put differently, it describes the share of all individuals in the population obtaining that outcome). For two probability mass functions (i.e., population distributions) f and g, the function f lower orthant dominates g if

(1) the cumulative probability mass at f is smaller than or equal to that at g for every lower hyperrectangle.[4]

Until quite recently, a characterization based on elementary operations (i.e., conditions specifying exactly which simple changes in a distribution are allowed to obtain another distribution which dominates it) has been missing.[5]

This and a related gap were recently addressed and partially filled by Meyer and Strulovici (2010, 2015) and Müller (2013). Indeed, for the bivariate case the former authors showed that one probability mass function *supermodular dominates* another if and only if the former probability mass function can be obtained from the latter probability mass function by increasing probability mass transfers and correlation-increasing switches. An increasing probability mass transfer is simply a shift of mass from a worse to a better outcome (i.e., such a transfer is a welfare improvement).[6] A correlation-increasing switch consists of two simultaneous transfers that move mass from intermediate outcomes to more extreme outcomes without changing the marginal distributions. For example, Tchen (1980), Epstein and Tanny (1980), Atkinson and Bourguignon (1982), Tsui (1999), Decancq (2012), and Sonne-Schmidt et al. (2016) argue that correlation-increasing switches are operations that increase inequality. Lower orthant dominance may be expressed with the help of diminishing transfers (reducing welfare) and correlation-increasing switches (increasing inequality). More precisely, Müller's (2013) characterization of lower orthant dominance for the general multivariate case directly implies, for the bivariate case, that (1) is equivalent to

(2) a finite sequence of diminishing bilateral transfers and correlation-increasing switches exists such that g can be obtained by f and where each intermediate transformation leads to a distribution.

In welfare terms, supermodular dominance corresponds to a population that is better off but the inequality is *higher*, whereas lower orthant dominance corresponds to a population that is better off and the inequality is *lower*, i.e., the latter concept provides a basis for making inequality-averse social welfare comparisons.

The approach by Meyer and Strulovici (2010, 2015) is constructive, but it is not shown that a distribution can be obtained after each elementary operation. In contrast, Müller (2013) shows the existence of such sequences, where a distribution is obtained after each elementary operation, but an explicit construction is not given.

The main contribution of this paper is to provide a constructive proof of the equivalence between (1) and (2). The proof yields an algorithm that returns a set of diminishing transfers and correlation-increasing switches whenever a lower orthant dominance relationship exists. The algorithm has quadratic time complexity in the number of outcomes. We also mention that the results for upper orthant dominance would be similar.

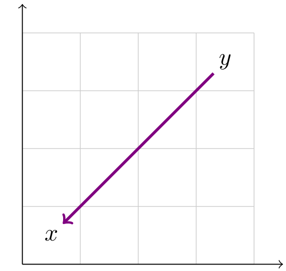

Fig. 2.1. Diminishing transfer.

2. Basics

Let $n, m \in \mathbb{N}$. For $x, y \in \mathbb{R}^m$, $x \leqslant (\geqslant)\, y$ denotes $x_i \leqslant (\geqslant)\, y_i$ for all $i = 1, \ldots, m$, and $x < (>)\, y$ means $x \leqslant (\geqslant)\, y$ and $x \neq y$. Similarly, for two functions $f, g : D \to \mathbb{R}$ on an arbitrary domain D, we write $f \geqslant (\leqslant)\, g$ if $f(x) \geqslant (\leqslant)\, g(x)$ for all $x \in D$, and $f > (<)\, g$ if $f \geqslant (\leqslant)\, g$ and $f \neq g$.

Denote $X(n, m) = X = \{x \in \mathbb{N}^2 \mid x \leqslant (n, m)\}$ the rectangle of size $n \times m$ and $\mathcal{F}(n, m) = \mathcal{F} = \{f : X \to \mathbb{R}_+\}$ be the set of all real-valued non-negative functions on the domain X. For $\emptyset \neq Y \subseteq X$ let $\max Y = y \in X$ be the componentwise maximum defined by $y_i = \max\{x_i \mid x \in Y\}$ for $i = 1{,}2$, and let $\min Y$ be the componentwise minimum defined analogously. Moreover, for $x \in X$, we denote the lower set $\downarrow x = \{y \in X \mid y \leqslant x\}$ as all elements of X having no component larger than the components of x.

In this paper we will use two fundamental operations. The first operation is a so-called diminishing transfer,[7] while the second is a correlation-increasing switch. For $f, g \in \mathcal{F}$ we say that g results from f

- by a *diminishing (bilateral) transfer* if there exist $x, y \in X$ such that $x < y$, $g(x) - f(x) = f(y) - g(y) > 0$, and $g(z) = f(z)$ for all $z \in X \setminus \{x, y\}$ (the underlying transfer is a transfer *from* y *to* x of size $\varepsilon = g(x) - f(x)$) and we use the notation $g = f_\varepsilon^{x \leftarrow y}$;
- by a *correlation-increasing switch* if there exist $x, y \in X$ such that $f(x) - g(x) = f(y) - g(y) = g(v) - f(v) = g(w) - f(w) > 0$ and $f(z) = g(z)$ for all $z \in X \setminus \{x, y, v, w\}$, where $v = \min(\{x, y\})$ and $w = \max(\{x, y\})$ (note that in this case x and y are incomparable; i.e., $x \nleqslant y \nleqslant x$, and that the underlying switch transfers $\varepsilon = f(x) - g(x)$ from each x and y to each v and w) and we use the notation $g = f_\varepsilon^{x \leftrightarrows y}$.

Note that a diminishing transfer may be represented as a composition of even more elementary transfers, where one only transfers mass "horizontally" and the other only "vertically". However, we use the current "composite" transfer because it is intuitive and simple and its decomposition into the mentioned elementary transfers is straightforward. A diminishing transfer is illustrated in Fig. 2.1, where mass is transferred from y to x. It should be noted that a diminishing transfer can be decomposed into a sequence of unit diminishing transfers from $y = (y_1, y_2)$ to either $(y_1 - 1, y_2)$ or $(y_1, y_2 - 1)$. This decomposition is, however, not unique. A correlation-increasing switch is illustrated in Fig. 2.2. As illustrated, mass is transferred from x to v and a similar mass is transferred from y to w. A symmetric transfer exists where mass is transferred from x to w and the same mass is transferred from y to v.

[4] In the continuous bivariate case, it is well-known (Atkinson and Bourguignon, 1982) that (1) holds if and only if the average utility of f is at least as high as that of g for any non-decreasing utility function with negative cross derivative.

[5] For example, Moyes (2012) points out in his Footnote 13 that such characterization is missing, even though there are results in the literature that are making steps in this direction.

[6] Indeed, the usual stochastic order is completely characterized by such transfers, as shown by, for example, Strassen (1965) and Kamae et al. (1977).

[7] Note that the term diminishing transfer has been used with another meaning by Lambert (2001, p. 62). We stick to its current meaning in order to be consistent with Range and Østerdal (2015).