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Null preference and the resolution of the topological social choice paradox

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HIGHLIGHTS

- Considered the role of null preference in social choice with non-Hausdorff topology of preference space.
- Constructed a contractible space from non-contractible one by adding the null preference point — the "null closure" procedure.
- Resolved the topological social choice paradox by proving an extension of Resolution Theorem.

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ABSTRACT

We investigate the role of contractibility in topological social choice theory. The Resolution Theorem states that there exists an aggregation map that is anonymous, unanimous, and continuous if and only if the space of individual preferences is contractible. Here, we turn a non-contractible space of social preferences (modeled as a CW complex) into a contractible space by adding the null preference which models full indifference of society, following the possibility results of Jones et al. (2003) which is based on a topology first considered but rejected by Le Breton and Uriarte (1990). We prove the corresponding extension of the Resolution Theorem by showing that the null preference as a social outcome precisely captures those voter profiles that represent a "tie" under a Chichilnisky map. Further, the space of these tie profiles is shown to have measure zero in the case that the space of individual preferences is a sphere in any dimension.

1. Introduction

Social choice theory aims to understand the nature of, and provide methods for, the aggregation of individual preferences to yield a social preference which is fair and satisfactory to individual voters on an axiomatic ground. An important example of this is in popular elections, where a large population must choose its favorite candidates. Where Arrow (1963) proved the nonexistence of rational choice for a finite set of alternatives, Chichilnisky and Heal (1983) (following Eckmann, 1954) set up a local differentiable framework on a n-dimensional cube of alternatives and proved that social welfare functions which are continuous, anonymous, and unanimous exist if and only if the space of preferences is contractible. For overviews, we refer the reader to Lauwers (2000) and Baigent (2011).

In this framework, the space of preferences is always assumed to be a CW complex. It is worth noting that the restriction to CW complexes is strong, but still leaves a very wide collection of possible spaces. Unfortunately, there are several noteworthy instances in previous literature where this requirement is ignored. It is incorrect to claim that any preference space has a Chichilnisky map if and only if it is contractible. It is crucial to recall that the theorem only applies to (parafinite) CW complexes.

Le Breton and Uriarte (1990) considered the idea of inserting a null preference for the special case of a preference space given by S^n, the n-sphere, which is a non-contractible finite CW complex. There, the new space $S^n \cup \{\mathcal{O}\}$ is allowed a special non-Hausdorff topology, where $\mathcal{O}$ represents a null preference point, a state of total indifferent to all candidates. However, this topology was quickly rejected by these authors because "it violated Hausdorff's separation axiom" (p. 134). Jones et al. (2003) seriously took up this idea and explored the existence of Chichilnisky maps when individual choices and/or the social outcome are allowed to have null preference $\mathcal{O}$. They were able to derive possibility results for the case of preference space modeled by S^n, where Resolution Theorem states that no Chichilnisky map could exist.

In this paper, we will describe a construction to insert a null preference $\mathcal{O}$ into any preference space P modeled as finite CW complexes rather than S^n. This new space $\tilde{P}$ is no longer a CW

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complex, but nonetheless $\tilde{P}$ is contractible. As contractibility is central to the Resolution Theorem, we will investigate here how a generalized construction of the "null closure" for any preference space P will affect the existence of Chichilnisky aggregation maps, and hence the resolution of topological social choice paradox.

After applying some tools from algebraic topology, we will be able to specify an output for profiles which contain votes that all "cancel out", and thus the problem is essentially reduced to the contractible case. Along the way, we will see that aggregation maps allowing individuals to have a null preference are not viable. We will also see that the existence of profiles that result in a "tie" is equivalent to a preference space being non-contractible. Finally, in the special case of $P = S^n$, we will show that the space of tie profiles is measure zero.

2. Contractibility and null preference in topological social choice

Let (X, τ) be any topological space. Define the space $(\tilde{X}, \tilde{\tau})$ by $\tilde{X} = X \cup \{\mathcal{O}\}$, where $\mathcal{O}$ is a new point, endowed with the topology $\tilde{\tau} = \tau \cup \{\tilde{X}\}$. That is, the only open neighborhood of $\mathcal{O}$ is the entire space. Intuitively, this means that the point $\mathcal{O}$ is "infinitely near" every other point of $\tilde{P}$. More precisely, the space $\tilde{X}$ is non-Hausdorff, and $\mathcal{O}$ is not separated from any other point in the sense that no neighborhood of $\mathcal{O}$ excludes any other point, and moreover every closed set in $\tilde{P}$ contains the point $\mathcal{O}$. This new space, which essentially follows the construction for S^n described in Le Breton and Uriarte (1990) and explored in length in Jones et al. (2003), has several useful properties, notably the fact that it has a non-empty specialization pre-order merely due to $\mathcal{O}$ as a common specialization of every point of X.

Lemma 2.1. *For any topological space X, the space $\tilde{X}$ is contractible.*

Proof. Define the map $F : \tilde{X} \times [0, 1] \to \tilde{X}$ by

$$F(x, t) = \begin{cases} x & \text{if } t < 1 \\ \mathcal{O} & \text{if } t = 1. \end{cases}$$

To see why this is continuous, note that there are two cases for an open set U in the image space. First, if $\mathcal{O} \notin U$, then the preimage is precisely $U \times [0, 1)$, which is open. The only remaining open subset of the image space is the entire range, which has the entire domain as preimage. Hence this is a continuous map, which defines a homotopy from the identity map to a constant map. □

Definition 2.2. Define the *null closure* of X to be the space $\tilde{X}$ obtained via the above construction.

In general, extending the preference space by adding a null preference requires the addition of one point and some open set (s) to the topology. Besides defining $\tilde{\tau}$ as discussed in the last subsection, the only other symmetrical way to define the topology on $P \cup \{\mathcal{O}\}$ would be to make $\mathcal{O}$ an isolated point, i.e. to make $\{\mathcal{O}\}$ an open set. However, this is then a non-contractible CW complex, for which the Resolution Theorem states that there are no Chichilnisky maps.

Le Breton and Uriarte (1990) "rejected" (p. 136) the null closure topology for $S^n \cup \{\mathcal{O}\}$ on the grounds that it does not satisfy the Hausdorff separation axiom. However, we think Hausdorff separability (which essentially turns a preference space into a metric space) is way too strong a requirement for social choice theories. We note that the null closure topology on $S^n \cup \{\mathcal{O}\}$, as the space of admissible preferences and/or social outcomes, only perturbed the Hausdorff topology on S^n "slightly", by having $\mathcal{O}$ as the only and common specialization point to all points of S^n—this is the only source of non-empty specialization order (and hence non-T_1 separability which precedes Hausdorff separability). This null closure construction augments the space of social outcomes to allow for Chichilnisky maps on any preference space, and has useful interpretations in applications, so we do not find this lack of Hausdorff separation to be a significant flaw. All Chichilnisky maps $P^n \to P$ are still Chichilnisky for $P^n \to \tilde{P}$.

3. Continuity and the null closure

With the construction of $\tilde{P}$, one next asks where it is appropriate to use $\tilde{P}$ or P. In Theorem 3.3, we use $\tilde{P}$ only as the range space of the aggregation map, and not as the space of possible preferences. It turns out that the choice made in Theorem 3.3 is the only one which allows for nontrivial aggregation maps. To show this, we take inspiration from Jones et al. (2003), where similar results are shown for the special case of $P = S^n$. We will follow a similar proof method, except that instead of using the metric structure on S^n, we will only need that the CW complex P is Hausdorff.

Proposition 3.1. *Let $f : (\tilde{P})^k \to \tilde{P}$ be continuous. Given any individual j and any profile $\mathbf{p}_{-j}$ for the remaining $k - 1$ voters, define the component map $\iota_j = f(\mathbf{p}_{-j}, \cdot) : \tilde{P} \to \tilde{P}$. Either $\iota_j(\mathcal{O}) = \mathcal{O}$ or else ι_j is constant.*

Proof. Consider $q = f(p_{-j}, \mathcal{O})$ and assume $q \neq \mathcal{O}$. Let $U \subseteq P$ be any open neighborhood of q. The preimage $(\iota_j)^{-1}(U)$ contains $\mathcal{O}$ and is open in $\tilde{P}$, and thus must be equal to all of $\tilde{P}$. Therefore $\mathrm{Im}(\iota_j) \subseteq U$.

Since U is arbitrary, we must have that $\mathrm{Im}(\iota_j)$ is contained in every open set containing q. But P is Hausdorff, so for any $p \in P$ with $p \neq q$, there will be some neighborhood of q which does not contain p. Hence the above intersection will be a singleton set $\{q\}$, and so ι_j is constant. □

Proposition 3.2. *The only continuous maps $f : (\tilde{P})^k \to P$ are the constant maps.*

Proof. First, note that $f(\bar{\mathcal{O}}) \in P$, where $\bar{\mathcal{O}}$ refers to the profile $(\mathcal{O}, \ldots, \mathcal{O}) \in (\tilde{P})^k$. Hence for any open neighborhood of $f(\bar{\mathcal{O}})$, the preimage is an open subset of $(\tilde{P})^k$ containing $\bar{\mathcal{O}}$.

It is immediate from the definitions that the only open subset of $(\tilde{P})^k$ containing $\bar{\mathcal{O}}$ is the entire space.

Since the preimage of any neighborhood U of $f(\bar{\mathcal{O}})$ contains all of $(\tilde{P}^k)$, we have $\mathrm{Im}(f) \subseteq U$, where $U \subseteq P$ was arbitrary. By the same intersection argument as above, we have $\mathrm{Im}(f) = \{f(\mathcal{O})\}$, hence f is constant. □

To summarize, if we allow individual preferences to be null, each individual either forces the null outcome with their null vote, or has no effect whatsoever. These two results show that it is only reasonable to use the null closure in the range, i.e. the space of social preferences. In particular, no voter may choose the null preference, but the outcome may be the null preference.

Now, we may state precisely our extension of the Resolution Theorem.

Theorem 3.3. *Let P be a preference space realized as a parafinite CW complex, and $\tilde{P}$ the null closure. Then for all $k \geq 1$ there exists a Chichilnisky map $\Phi : P^k \to \tilde{P}$ such that the preimage of $\mathcal{O}$ is empty if and only if P is contractible.*

The proof is given in Appendix A. For a Chichilnisky map $\Phi : P^k \to \tilde{P}$, we will call $\Phi^{-1}(\{\mathcal{O}\})$ the set of *tie profiles*. The main idea of the proof, of course, is to construct a Chichilnisky map $P^k \to \tilde{P}$ for any preference space P and any $k \geq 1$. Our construction is similar to that in Chichilnisky and Heal (1983), with an additional step to find tie profiles to send to $\mathcal{O}$. As previously mentioned, this