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Bankruptcy games with nontransferable utility

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HIGHLIGHTS

- This paper introduces a modified model for bankruptcy games with nontransferable utility.
- This paper shows that bankruptcy games are compromise stable and reasonable stable.
- This paper axiomatically characterizes the class of game theoretic bankruptcy rules by truncation invariance.

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ABSTRACT

This paper analyzes bankruptcy games with nontransferable utility as a generalization of bankruptcy games with monetary payoffs. Following the game theoretic approach to NTU-bankruptcy problems, we study some appropriate properties and the core of NTU-bankruptcy games. Generalizing the core cover and the reasonable set to the class of NTU-games, we show that NTU-bankruptcy games are compromise stable and reasonable stable. Moreover, we derive a necessary and sufficient condition for an NTU-bankruptcy rule to be game theoretic.

1. Introduction

A bankruptcy problem is an elementary allocation problem in which claimants have individual claims on an estate which cannot be satisfied together. Bankruptcy theory studies allocations of the estate among the claimants, taking into account the corresponding claims. In a bankruptcy problem with transferable utility (cf. O'Neill, 1982), the estate and claims are of a monetary nature. These problems are well-studied, both from an axiomatic perspective and a game theoretic perspective. We refer to Thomson (2003) for an extensive survey, to Thomson (2013) for recent advances, and to Thomson (2015) for an update.

Carpente et al. (2013) extended TU-bankruptcy problems by explicitly including individual but comparable utility functions on the domain of feasible monetary payoffs. Dietzenbacher et al. (2016) generalized monetary bankruptcy problems to bankruptcy problems with nontransferable utility in which individual utility is represented in incompatible measures. The estate can take a more general shape and corresponds to a set of feasible utility allocations. Dietzenbacher et al. (2016) analyzed these NTU-bankruptcy problems from an axiomatic perspective by formulating appropriate properties for bankruptcy rules and studying their implications. In particular, they focused on proportionality, equality, and duality in bankruptcy problems with nontransferable utility, which resulted in axiomatic characterizations of the proportional rule and the constrained relative equal awards rule. Dietzenbacher et al. (2017a) continued on this axiomatic approach by studying several consistency notions and formulating the relative adjustment principle.

Orshan et al. (2003) analyzed NTU-bankruptcy problems from a game theoretic perspective by introducing an associated NTU-bankruptcy game. Estévez-Fernández et al. (2014) pointed out that coalitions can attain payoff allocations outside the estate in this game, which contradicts the original idea of O'Neill (1982). They redefined NTU-bankruptcy games to stay in line with this original idea about TU-bankruptcy games, while focusing on convexity and compromise stability. However, it turns out that their NTU-bankruptcy game does not straightforwardly generalize the original TU-bankruptcy game, since the attainable payoff allocations of subcoalitions are explicitly bounded by individual claims.

This paper studies a slightly modified version of the model of Orshan et al. (2003) for NTU-bankruptcy games which both generalizes the model for TU-bankruptcy games and stays in line with the idea of O'Neill (1982). Focusing on the structure of the core, we analyze NTU-bankruptcy games along the lines of Curiel et al. (1987). They showed that TU-bankruptcy games are convex, i.e. the core equals the Weber set, and compromise stable, i.e. the core equals the core cover. We introduce the notion of reasonable stability to describe games for which the core equals the reasonable set. We show that reasonable stability is equivalent to the combination of convexity and compromise stability on the class of TU-games, which means that TU-bankruptcy games are reasonable

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stable. Generalizing the core, the core cover, and the reasonable set to the class of NTU-games, we show that NTU-bankruptcy games are compromise stable and reasonable stable as well.

Curiel et al. (1987) also showed that a TU-bankruptcy rule is game theoretic if and only if it satisfies truncation invariance. This means that there exists a solution for TU-games which coincides on the class of bankruptcy games with a certain bankruptcy rule if and only if this bankruptcy rule satisfies truncation invariance. We generalize this characterization to rules for bankruptcy problems with nontransferable utility.

This paper is organized in the following way. Section 2 provides a formal overview of notions for transferable utility games and bankruptcy problems. Section 3 generalizes some notions for transferable utility games to the class of nonnegative games with nontransferable utility. Section 4 introduces and analyzes a modified model for bankruptcy games with nontransferable utility.

2. Preliminaries

2.1. Transferable utility games

Let N be a nonempty and finite set of *players*. An *order* of N is a bijection $\sigma : \{1, \dots, |N|\} \to N$. The set of all orders of N is denoted by $\Pi(N)$ and the set of all *coalitions* is denoted by $2^N = \{S \mid S \subseteq N\}$. A *transferable utility game* is a pair (N, v) in which $v : 2^N \to \mathbb{R}$ assigns to each coalition $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. Let TU^N denote the class of all transferable utility games with player set N. For convenience, a TU-game is denoted by $v \in \mathrm{TU}^N$.

Let $v \in \mathrm{TU}^N$. The *marginal vector* $M^\sigma(v) \in \mathbb{R}^N$ corresponding to $\sigma \in \Pi(N)$ is for all $k \in \{1, \dots, |N|\}$ given by

$$M^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), \dots, \sigma(k)\}) - v(\{\sigma(1), \dots, \sigma(k-1)\}).$$

The vector $K(v) \in \mathbb{R}^N$ is for all $i \in N$ given by

$$K_i(v) = v(N) - v(N \setminus \{i\}),$$

and the vector $k(v) \in \mathbb{R}^N$ is for all $i \in N$ given by

$$k_i(v) = \max_{S \in 2^N : i \in S} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} K_j(v) \right\}.$$

Let $v \in \mathrm{TU}^N$. The *core* is given by

$$\mathcal{C}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), \forall S \in 2^N : \sum_{i \in S} x_i \geq v(S) \right\},$$

the *Weber set* (cf. Weber, 1988) is given by

$$\mathcal{W}(v) = \mathrm{Conv} \left\{ M^\sigma(v) \;\middle|\; \sigma \in \Pi(N) \right\},$$

the *core cover* (cf. Tijs and Lipperts, 1982) is given by

$$\mathcal{CC}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), k(v) \leq x \leq K(v) \right\},$$

and the *reasonable set* (cf. Gerard-Varet and Zamir, 1987) is given by

$$\mathcal{R}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), \forall i \in N : \min_{\sigma \in \Pi(N)} M^\sigma_i(v) \leq x_i \leq \max_{\sigma \in \Pi(N)} M^\sigma_i(v) \right\}.$$

We have $\mathcal{C}(v) \subseteq \mathcal{W}(v) \subseteq \mathcal{R}(v)$ and $\mathcal{C}(v) \subseteq \mathcal{CC}(v) \subseteq \mathcal{R}(v)$. A TU-game $v \in \mathrm{TU}^N$ is called *convex* (cf. Shapley, 1971 and Ichiishi, 1981) if $\mathcal{C}(v) = \mathcal{W}(v)$, and *compromise stable* (cf. Quant et al., 2005) if $\mathcal{C}(v) = \mathcal{CC}(v)$ and $\mathcal{CC}(v) \neq \emptyset$.

We introduce the notion of reasonable stability to describe games for which the core and the reasonable set coincide. Moreover, we show that reasonable stability is equivalent to the combination of convexity and compromise stability.

Definition 2.1 (*Reasonable Stability*). A transferable utility game $v \in \mathrm{TU}^N$ is called *reasonable stable* if $\mathcal{C}(v) = \mathcal{R}(v)$.

Theorem 2.1. *A transferable utility game is reasonable stable if and only if it is convex and compromise stable.*

Proof. Assume that $v \in \mathrm{TU}^N$ is reasonable stable. Then we have $\mathcal{C}(v) = \mathcal{R}(v)$. Since $\mathcal{C}(v) \subseteq \mathcal{W}(v) \subseteq \mathcal{R}(v)$ and $\mathcal{C}(v) \subseteq \mathcal{CC}(v) \subseteq \mathcal{R}(v)$, this means that $\mathcal{C}(v) = \mathcal{W}(v)$ and $\mathcal{C}(v) = \mathcal{CC}(v)$. Hence, $v \in \mathrm{TU}^N$ is convex and compromise stable.

Assume that $v \in \mathrm{TU}^N$ is convex and compromise stable. Since $v \in \mathrm{TU}^N$ is convex, we have $\min_{\sigma \in \Pi(N)} M^\sigma_i(v) = v(\{i\})$ and $\max_{\sigma \in \Pi(N)} M^\sigma_i(v) = v(N) - v(N \setminus \{i\})$ for all $i \in N$. Moreover, we have $k_i(v) = v(\{i\})$ for all $i \in N$. This means that $\min_{\sigma \in \Pi(N)} M^\sigma_i(v) = k_i(v)$ and $\max_{\sigma \in \Pi(N)} M^\sigma_i(v) = K_i(v)$ for all $i \in N$, so $\mathcal{CC}(v) = \mathcal{R}(v)$. Since $v \in \mathrm{TU}^N$ is compromise stable, this implies that $\mathcal{C}(v) = \mathcal{CC}(v) = \mathcal{R}(v)$. Hence, $v \in \mathrm{TU}^N$ is reasonable stable. □

2.2. Bankruptcy problems

Let N be a nonempty and finite set of *claimants*. A *bankruptcy problem with transferable utility* (cf. O'Neill, 1982) is a triple (N, M, c) in which $M \in \mathbb{R}_+$ is an *estate* and $c \in \mathbb{R}^N_+$ is a vector of *claims* of N on M for which $\sum_{i \in N} c_i \geq M$. Let TUBR^N denote the class of all bankruptcy problems with transferable utility with claimant set N. For convenience, a TU-bankruptcy problem is denoted by $(M, c) \in \mathrm{TUBR}^N$.

For any set of payoff allocations $E \subseteq \mathbb{R}^N_+$,

- the *nonnegative comprehensive hull* is given by $\mathrm{comp}(E) = \{x \in \mathbb{R}^N_+ \mid \exists y \in E : y \geq x\}$;
- the *strong Pareto set* is given by $\mathrm{SP}(E) = \{x \in E \mid \neg\exists y \in E, y \neq x : y \geq x\}$;
- the *strong upper contour set* is given by $\mathrm{SUC}(E) = \{x \in \mathbb{R}^N_+ \mid \neg\exists y \in E, y \neq x : y \geq x\}$;
- the *weak upper contour set* is given by $\mathrm{WUC}(E) = \{x \in E \mid \neg\exists y \in E : y > x\}$.

Note that $\mathrm{SP}(E) \subseteq \mathrm{SUC}(E) \subseteq \mathrm{WUC}(E)$. A set of payoff allocations $E \subseteq \mathbb{R}^N_+$ is called *(nonnegative) comprehensive* if $E = \mathrm{comp}(E)$, and called *nonleveled* if $\mathrm{SUC}(E) = \mathrm{WUC}(E)$.

A *bankruptcy problem with nontransferable utility* (cf. Dietzenbacher et al., 2016) is a triple (N, E, c) in which $E \subseteq \mathbb{R}^N_+$ is a nonempty, closed, bounded, comprehensive and nonleveled *estate* and $c \in \mathrm{SUC}(E)$ is a vector of *claims* of N on E. Let BR^N denote the class of all bankruptcy problems with nontransferable utility with claimant set N. For convenience, an NTU-bankruptcy problem is denoted by $(E, c) \in \mathrm{BR}^N$. Note that any TU-bankruptcy problem $(M, c) \in \mathrm{TUBR}^N$ gives rise to the NTU-bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \leq M\}$.

Let $(E, c) \in \mathrm{BR}^N$. The vector of *utopia values* $u^E \in \mathbb{R}^N_+$ is for all $i \in N$ given by

$$u^E_i = \max\{x_i \mid x \in E\}.$$

The vector of *truncated claims* $\hat{c}^E \in \mathbb{R}^N_+$ is for all $i \in N$ given by

$$\hat{c}^E_i = \min\{c_i, u^E_i\}.$$

The vector of *minimal rights* $m(E, c) \in \mathbb{R}^N_+$ is for all $i \in N$ given by

$$m_i(E, c) = \begin{cases} \max\{x \mid (x, c_{N\setminus\{i\}}) \in E\} & \text{if } (0, c_{N\setminus\{i\}}) \in E; \\ 0 & \text{if } (0, c_{N\setminus\{i\}}) \notin E. \end{cases}$$

Note that $m(E, c) \in E$, $\hat{c}^E \in \mathrm{SUC}(E)$, and $m(E, c) \leq \hat{c}^E$.