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Graphs and (levels of) cooperation in games: Two ways how to allocate the surplus

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HIGHLIGHTS

- We study games where cooperation is restricted by two exogenous structures.
- We define two values that generalize the Shapley, Banzhaf, Owen, and Myerson values.
- We provide a characterization for each of the values by means of properties.
- Some of these properties are logically comparable.

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ABSTRACT

We analyze surplus allocation problems where cooperation between agents is restricted both by a communication graph and by a sequence of embedded partitions of the agent set. For this type of problem, we define and characterize two new values extending the Shapley value and the Banzhaf value, respectively. Our results enable the axiomatic comparison between the two values and provide some basic insights for the analysis of fair resource allocation in today's fully integrated societies.

1. Introduction

As a general rule, the layout of political administrations consists of a series of embedded layers. The EU, for instance, is integrated by countries, which are organized in regions, which, in turn, are divided into smaller administrative units, say cities, and so on. As far as his/her relation with political institutions – and hence with political power – is concerned, a EU citizen can typically deal with one institution only at each level. The overall political and administrative layout typically affects the possibilities of citizens to cooperate: there would be no barriers to cooperation if no political institution existed. For example, it is generally easier to engage in a business venture within a common labor market under a single set of rules than across several markets, each regulated by a different set of rules.[1]

The fact that citizens are integrated in a sequence of embedded layers does not prevent them from interacting in various other ways, some of which go beyond the lines set up by political institutions. For instance, there is a great flow of workers between the major financial markets even though they do not constitute a common political entity. This is possible due to the existence of certain networks, very especially transportation and communication networks. More generally, trade between individuals and firms does not necessarily submit to political structures, especially after the outset of globalization. Rivers, highways, and train networks, among others, facilitate business and enable individuals and firms to use their full potential. The production chains of some major corporations, in particular, are now distributed over several countries, thanks to the existence of many global networks. With such non-political linkages boosting economic growth, most of the redistribution policies – say, taxes and public expenditures – are, however, decided only by political institutions operating at national or regional level.

In a framework where production is not only intertwined with political institutions but also with a host of other (binary) linkages such as social networks, is there a reasonable way to address the issue of how the aggregate spoils generated by all citizens should be shared? Providing knowledge about solutions to this problem is the general object of this paper. Taking here mainly a normative approach, we focus on certain properties that may be required for surplus allocation rules. In this vein, it is worth noting that the increase in inequality (see e.g. Atkinson, 2015) that has taken place in the past few years in many countries has raised a major concern:

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[1] The so-called *border effect* is a well-documented phenomenon in international trade (see e.g. Evans, 2003).

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policy decisions should be adopted to guarantee that all citizens benefit from globalization. Our analysis features some elements of fairness that revolve around this concern.

To elaborate, we analyze the class of surplus allocation problems in which cooperation between agents – say, citizens – is restricted both by a sequence of embedded partitions of the agent set and by a communication graph between agents. Formally, we consider a triple made up of a cooperative game, a levels structure, and a non-directed graph, which we call a *Game with Graph-Restricted Communication and Levels Structure of Cooperation*. First, the cooperative game describes the potential gains that any subset of players can attain on their own, assuming that cooperation is unrestricted. Second, the sequence of embedded partitions of the player set (the so-called *levels structure*) represents the different "administrative units" in which players are organized. These given arrangements among agents restrict or hinder the formation of coalitions when some of its members belong to different units at some levels. Third and last, the *communication graph* accounts for the bilateral relations that may exist between players, e.g. due to commercial relations. We assume that a coalition of agents can cooperate only if all of its members are path-connected within the graph. Such links extend naturally to higher units: A city is connected with another city if there is a link between their citizens. As it is the case with any levels structure, a communication graph affects cooperation between all players, whether directly or indirectly.

Games with graph-restricted communication and levels structure of cooperation provide an appropriate model to address the normative problem of how to allocate the surplus that all the agents can potentially create under the restrictions placed by all layers of political institutions and all communication networks. To make progress in the analysis of this problem, we introduce two values (or point-valued solutions) for games with graph-restricted communication and levels structure of cooperation. The first value focuses on the orderings in which coalitions are formed, while the second value considers coalitions directly, without any reference to how they are formed. As is standard in the literature, the first value extends the Shapley value and the second value extends the Banzhaf value, the two classic solutions for cooperative games. We then provide a characterization of each value by means of several properties (or axioms). Such properties are of two types. A first type describes particular ways how the surplus sharing should be affected by alterations of the communication graph. A second set of properties deals with changes in the levels structure. Importantly, the latter properties used in either characterization result are (logically) comparable. This fact may be useful when asking whether to use one value or the other for a particular situation.[2]

Our contribution belongs to the extensive literature of games with restricted cooperation, which dates back at least to Aumann and Drèze (1974) (see also Owen, 1977; Myerson, 1977). Owen (1977) considers situations in which cooperation is restricted by a partition of the player set – the so-called *games with coalition structure* –, which are particular instances of the more general *games with coalition configuration* (see Albizuri et al., 2006; Albizuri and Aurrekoetxea, 2006; Andjiga and Courtin, 2015). In the latter model, coalitions are not necessarily disjoint. In turn, Myerson (1977) considers situations in which cooperation is restricted by a communication graph, the so-called *games with graph-restricted communication*. Several further papers have built on these models or extended them (see e.g. Owen, 1986; Amer et al., 2002; Alonso-Meijide and Fiestras-Janeiro, 2002, 2006). Singularly, Winter (1989) generalized the model of games with coalition structure to include restrictions to cooperation that may exist at various levels. He refers to his extended framework as *games with levels structure (of cooperation)*. Both types of restrictions to cooperation, i.e. levels structures and undirected graphs, can, however, exist simultaneously. To account for this possibility, Vázquez-Brage et al. (1996) and Alonso-Meijide et al. (2009) have already proposed and characterized generalizations of the Shapley value and the Banzhaf value for games with both a coalition structure and a communication graph. The model we analyze is a natural generalization of the latter, insofar as it considers a levels structure instead of coalition structure (i.e, a levels structure with a single level).

The paper is organized as follows: In Section 2 we set the notation and introduce the main concepts from the literature. In Section 3 we define two new values for games with graph-restricted communication and levels structure of cooperation, which we characterize by a number of properties and then compare. Section 4 concludes. The proofs are given in the Appendix.

2. Notations and preliminaries

2.1. Cooperative games with transferable utility

Let Ω denote the (possibly infinite) set of potential players. A *cooperative game with transferable utility* (or simply a *game*) is a pair (N, v), where $\emptyset \neq N \subseteq \Omega$ is a finite set of players and $v : 2^N = \{S : S \subseteq N\} \to \mathbb{R}$ is the *characteristic function*, with $v(\emptyset) = 0$. For every coalition $S \subseteq N$, $v(S)$ represents the worth of coalition S, i.e., the total payoff that members of the coalition can obtain by agreeing to cooperate. We denote the collection of all games by $\mathcal{G}$. For the sake of readability, we henceforth abuse notation slightly and write $T \cup i$ and $T \setminus i$ instead of $T \cup \{i\}$ and $T \setminus \{i\}$ for $T \subseteq N$ and $i \in N$, respectively. We use the operator $|\cdot|$ to denote the cardinality of a finite set.

A *value on* $\mathcal{G}$ is a map, f, that assigns a unique vector $\mathsf{f}(N, v) \in \mathbb{R}^N$ to every $(N, v) \in \mathcal{G}$. A *permutation* of N is a bijective map $\pi : N \to N$. Let $\Pi(N)$ denote the set of permutations of N. Given $\pi \in \Pi(N)$ and $i \in N$, let $\pi^{-1}[i]$ indicate the set of players ordered before i in permutation π, i.e., $\pi^{-1}[i] = \{j \in N : \pi(j) < \pi(i)\}$. Next, we present the formal definitions of two well-known values on $\mathcal{G}$, namely the Shapley and Banzhaf values.

Definition 2.1. The *Shapley value* (Shapley, 1953), Sh, is the value on $\mathcal{G}$ defined for every $(N, v) \in \mathcal{G}$ and $i \in N$ by

$$\mathsf{Sh}_i(N, v) = \frac{1}{|\Pi(N)|} \sum_{\pi \in \Pi(N)} \left[v\left(\pi^{-1}[i] \cup i\right) - v\left(\pi^{-1}[i]\right)\right].$$

The *Banzhaf value* (Banzhaf, 1965), Ba, is the value on $\mathcal{G}$ defined for every $(N, v) \in \mathcal{G}$ and $i \in N$ by

$$\mathsf{Ba}_i(N, v) = \frac{1}{2^{|N|-1}} \sum_{S \subseteq N \setminus i} [v(S \cup i) - v(S)].$$

The differences between these values are well known from an axiomatic viewpoint (see e.g. Young, 1985; Feltkamp, 1995; Nowak, 1997). We note that the Shapley value can be alternatively defined as

$$\mathsf{Sh}_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{|S|!|N \setminus (S \cup i)|!}{|N|!} [v(S \cup i) - v(S)].$$

2.2. Games with graph-restricted communication

A *communication graph* is an undirected graph without loops defined on a finite set of nodes. That is, (N, C) is a communication graph if N is a finite set of nodes and C is a set of links between the nodes. A *link* between i and j is denoted by $\{i : j\}$ (note that $\{i : j\} = \{j : i\}$). Given $i, j \in S \subseteq N$, we say that i and j are *path-connected* (or just *connected*) in S by C if there is a *path* in S connecting them,

[2] We elaborate on this issue in Section 3.3.