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All-pay contests with performance spillovers☆

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HIGHLIGHTS

- We generalize the results of Siegel (2009, 2010) to accommodate performance spillovers.
- We prove that an all-pay contest with additively separable spillovers has a unique Nash equilibrium.
- We construct the equilibrium payoffs and strategies in the all-pay contests with additively separable spillovers.
- We also construct the unique Nash equilibrium in a two-player all-pay contest with multiplicatively separable spillovers.

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ABSTRACT

This paper generalizes the results of Siegel (2009, 2010) to accommodate performance spillovers, with which a player's performance in a contest may affect the performance cost of another player. More precisely, we show that, if for any player, the spillovers from other players' performance enter his cost in an additively separable form, then an all-pay contest has a unique Nash equilibrium. Moreover, we construct the equilibrium payoffs and strategies. Both the equilibrium uniqueness and construction are generalized to multiplicatively separable spillovers in a two-player contest.

1. Introduction

Performance spillovers are prevalent in contest situations. For example, higher expenditure from a lobbyist may make it easier for another lobbyist to justify his expenditure; a company's R&D effort may benefit its rivals, and hard working classmates make it easier, or less costly, for an individual student to study hard. Siegel (2009, 2010) studies contests among asymmetric players without spillovers, and Baye et al. (2012) study contests between two symmetric players with spillovers. The two setups demonstrate different equilibrium properties. For example, an asymmetric contest without spillovers has a unique Nash equilibrium, while a symmetric contest with spillovers may have one or more Nash equilibria depending on parameter values. To bridge the gap between these studies, this paper investigates contests that allow spillovers among asymmetric players.

Specifically, we introduce two types of spillovers in contests: additive and multiplicative. With additive spillovers, the other players' performance levels enter a player's cost function in an additively separable way. For example, given the other player's performance s_j, player i's cost of performance s_i is $C_i(s_i, s_j) = s_i - \bar{s}$, which means player i's cost depends on not only his own performance but also the average performance $\bar{s} = (s_i + s_j)/2$. As a result, player j's performance affects i's cost through the average performance.[1] This is an aggregate game with linear structure. Linear models with aggregate performance are widely used in empirical studies of spillovers in innovation (e.g. Audretsch and Feldman, 1996), workplaces (e.g. Mas and Moretti, 2009) and education (e.g. Angrist, 2014). Acemoglu and Jensen (2013) provide a theoretic study on spillovers through the average or aggregate action in more general competitions. These studies focus on competitions that are not based on performance ranking, so those competitions are different from contests.

With multiplicative spillovers, the other players' performance levels affect one player's performance cost in a multiplicatively separable way. An example of such cost functions is $C_i(s_i, s_j) = \bar{s}s_i$, which means the average performance $\bar{s}$ affects the marginal cost of player i's performance s_i.[2] Production functions with such a multiplicative form are used in studies of spillovers in R&D (e.g. Griliches, 1991) to capture the aggregate knowledge's effect on an individual firm's marginal productivity. They are also used in studies of more general social interactions, e.g., Glaeser and Sacerdote (2003).

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1 See more in Example 1.

2 See more in Example 2.

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If we introduce performance spillovers into a contest, the original equilibrium strategies may no longer be an equilibrium.[3] However, we show that all-pay contests with additive or multiplicative spillovers have a unique Nash equilibrium. Moreover, we manage to construct the equilibrium payoffs and strategies. Both equilibrium uniqueness and characterization are useful for applications involving contest design in the presence of spillovers.

2. Additive spillovers

Our model builds on that of Siegel (2010), to which we add the possibility of performance spillovers. Consider a contest in which n risk neutral players compete for m homogeneous monetary prizes, where $0 < m < n$.[4] The prize value is normalized to 1.[5] Denote the set of players as $N = \{1, \ldots, n\}$. Each player i simultaneously chooses a performance level, or score, $s_i \geq 0$. Let $\mathbf{s} = (s_i)_{i \in N}$ be the scores of all players, and $\mathbf{s}_{-i} = (s_j)_{j \in N\setminus\{i\}}$ be the scores of all players except i. Given all players' scores $\mathbf{s}$, player i's payoff is $u_i(\mathbf{s}) = P_i(\mathbf{s}) - C_i(\mathbf{s})$, where $P_i : \mathbb{R}_+^n \to [0, 1]$ is player i's probability of winning, and $C_i : \mathbb{R}_+^n \to \mathbb{R}$ is his cost of score. Note whether he wins or not, player i incurs the cost.[6] The probability of winning is $P_i(\mathbf{s}) = 1$ if i's score s_i exceeds those of at least $n - m$ other players, $P_i(\mathbf{s}) = 0$ if s_i is lower than those of at least m others, and $P_i(\mathbf{s})$ equals any value in $[0, 1]$ otherwise. For each i, $C_i(\mathbf{s})$ is strictly increasing in s_i, meaning player i's score s_i is costly for him. Note that $C_i(\mathbf{s})$ depends on all players' scores, so there may be spillovers. If $C_i(\mathbf{s})$ is independent of $\mathbf{s}_{-i}$, there are no spillovers, and our setup reduces to that of Siegel (2010).

We assume that the spillovers from other players' scores enter the cost in an *additively separable* way, i.e., $C_i(\mathbf{s}) = K_i(s_i) + H_i(\mathbf{s}_{-i})$ for each i, where $K_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $H_i : \mathbb{R}_+^{n-1} \to \mathbb{R}$ may differ among players, representing asymmetry in costs and spillovers respectively. The contest is of complete information, so these functions are commonly known. Recall that $C_i(\mathbf{s})$ is strictly increasing in s_i and $H_i(\mathbf{s}_{-i})$ is independent of s_i, so $K_i(s_i)$ is also strictly increasing in s_i. Then, assume that there exists $s_{\max} > 0$ such that $K_i(s_{\max}) > 1$ for all i, and define player i's reach as $r_i = K_i^{-1}(1)$, and re-index the players such that $r_1 \geq \cdots \geq r_n$.[7] We assume $r_i \neq r_{m+1}$ for $i \neq m + 1$. In addition, assume $K_i(0) = 0$ and K_i is continuous and piecewise analytic on $[0, r_{m+1}]$.[8] Moreover, for each $j \neq i$, $H_i(\mathbf{s}_{-i})$ is piecewise continuous in s_j on $[0, r_{m+1}]$.[9] The above contest is referred to as the contest with additive spillover. The following example illustrates the general model in a linear setup.

Example 1. Suppose the cost is $C_i(\mathbf{s}) = c_i s_i - h\bar{s}$, where $c_i \in \mathbb{R}_+$ is player i's marginal cost of score, and $\bar{s} = (\sum_{i=1}^n s_i)/n$ is the average score. Here the spillover depends on the average score, and h measures the scale of spillover. If $h = 0$, there is no spillover. If h is positive (negative), a higher average score makes player i's score less (more) costly. Assume distinct marginal costs so that $0 < c_1 < \cdots < c_n$.[10] In this example, $K_i(s_i) = (c_i - h/n)s_i$ and $H_i(\mathbf{s}_{-i}) = -h(\sum_{j \neq i} s_j)/n$. The assumption $\partial C_i(\mathbf{s})/\partial s_i > 0$ requires $h < nc_i$ for all i.[11] Both functions depend on the spillover parameter h. If h is positive (negative), $K_i(s_i)$ is lower (higher) than player i's scoring cost $c_i s_i$.

A strategy profile constitutes a Nash equilibrium if each player's (mixed) strategy assigns a probability of one to the set of his best responses against the strategies of other players. We only consider Nash equilibria here.

Equilibrium characterization. In the absence of spillovers, the method of Siegel (2009) can be used to derive equilibrium payoffs, with which equilibrium strategies can be constructed according to the algorithm of Siegel (2010). However, this approach is not applicable here. This is because with spillovers, we can no longer derive equilibrium payoffs as in the case without spillovers.

In contrast to Siegel's method, our method first constructs equilibrium strategies, which we then use to derive equilibrium payoffs. Given the original contest, consider an auxiliary contest with the same prizes but different players, whose cost functions are $K_i(s_i)$ for all i. The auxiliary contest has no spillover, but it is different from the original contest without spillovers. For instance, if $h = 0$ in Example 1, there is no spillover, and a player's scoring cost is $c_i s_i$, which is different from $K_i(s_i) = (c_i - h/n)s_i$ in the auxiliary contest.

According to Siegel (2010), the auxiliary contest has a unique equilibrium. In this contest, let $G_i : \mathbb{R}_+ \to [0, 1]$ be the c.d.f. representing player i's equilibrium strategy, and $\mathbf{G} = (G_i)_{i \in N}$ be the equilibrium. If G_i assigns probability 1 to a single score, it represents a pure strategy.

Lemma 1 (*Strategic Equivalence*)**.** *A strategy profile is an equilibrium in the contest with additive spillovers if and only if it is an equilibrium in the auxiliary contest.*

Proof. In the auxiliary contest, if the other players use strategies $\mathbf{G}_{-i} = (G_j)_{j \in N\setminus\{i\}}$, player i's expected payoff from choosing s_i is $E[P_i(\mathbf{s}) - K_i(s_i)]$. In the contest with spillovers, if the others players use strategies $\mathbf{G}_{-i}$, player i's expected payoff from choosing s_i becomes $E[P_i(\mathbf{s}) - K_i(s_i)] - E[H_i(\mathbf{s}_{-i})]$, where $E[H_i(\mathbf{s}_{-i})] = \int H_i(\mathbf{s}_{-i}) d\mathbf{G}_{-i}(\mathbf{s}_{-i})$ is independent of his score.[12] The independence is a result of the additive separability. Thus, $\mathbf{G}$ is also an equilibrium in the contest with spillovers. Similarly, the converse is also true, i.e., any equilibrium in the contest with spillovers is also an equilibrium in the auxiliary contest. ■

The result below shows that the original contest with spillovers also has a unique equilibrium, and it is the same one constructed in the auxiliary contest.

Proposition 1. *The all-pay contest with additively spillovers has a unique equilibrium, which is the same as the one that the algorithm of* Siegel (2010) *constructs for the auxiliary contest.*

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3 See Examples 1 and 2.

4 Our results can be extended to heterogeneous prizes. For example, Bulow and Levin (2006) and González-Díaz and Siegel (2013) study contests with arithmetic prize sequences (with constant first order differences), and Xiao (2016) studies contests with quadratic prize sequences (with constant second order differences) or geometric prize sequences (with constant ratios between two consecutive prizes). Equilibrium uniqueness and construction are established in those contests. By the same argument in this paper, we can generalize those results to the case of additively separable spillovers.

5 Our analysis can be extended to allow players to have asymmetric valuations of the prize.

6 Because of the all-pay feature, the cost is sunk, so it remains the same whether a player wins. As a result, the spillovers represented by the cost functions also remain the same whether a player wins or not. In contrast, Baye et al. (2012) also consider rank-order spillovers that depend on the rank of a player's score, and demonstrate possibly multiple equilibria in the presence of rank-order spillovers.

7 The definition of "reach" is first introduced by Siegel (2009).

8 A function is piecewise analytic on an interval if the interval can be partitioned into a finite number of closed intervals such that the restriction of the function to each interval is analytic.

9 A function is piecewise continuous on an interval if the function is continuous on all points in the interval except a finite number of points at which the function has finite limits.

10 This is to ensure the assumption that $r_i \neq r_{m+1}$ for $i \neq m + 1$ is satisfied.

11 Otherwise, with $h > nc_i$, it is optimal for player i to choose $s_i = +\infty$.

12 According to Siegel (2010), each player j's equilibrium strategy G_j is continuous with a finite support. Moreover, H_i is piecewise continuous so is bounded over the supports of $\mathbf{G}_{-i}$. Hence, $E[H_i(\mathbf{s}_{-i})] = \int H_i(\mathbf{s}_{-i}) d\mathbf{G}_{-i}(\mathbf{s}_{-i}) < +\infty$.