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Continuity and completeness of strongly independent preorders[☆]

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HIGHLIGHTS

- Presents conflicts between continuity and completeness assumptions for preorders.
- The preorders are on possibly infinite dimensional convex sets.
- Applications to decision making under risk and uncertainty are given.

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ABSTRACT

We show that a strongly independent preorder on a possibly infinite dimensional convex set that satisfies two of the following conditions must satisfy the third: (i) the Archimedean continuity condition; (ii) mixture continuity; and (iii) comparability under the preorder is an equivalence relation. In addition, if the preorder is nontrivial (has nonempty asymmetric part) and satisfies two of the following conditions, it must satisfy the third: (i′) a modest strengthening of the Archimedean condition; (ii) mixture continuity; and (iii′) completeness. Applications to decision making under conditions of risk and uncertainty are provided, illustrating the relevance of infinite dimensionality.

1. Introduction and main results

The completeness axiom of expected utility has long been regarded as dubious, while the usual continuity axioms are typically seen as innocuous. However, given a strongly independent preorder on a convex set, we show that the standard Archimedean and mixture continuity axioms together imply that the possibilities for incompleteness are highly restricted, in a sense made precise below. In particular, they rule out the most natural preference structures for agents who find they cannot exactly compare two alternatives. If the Archimedean axiom is slightly strengthened in a natural direction, the room for incompleteness vanishes entirely: the preorder must be complete. The first claim strengthens a result of Aumann (1962), while the second extends (Dubra, 2011) from the finite to the infinite dimensional case. We shortly give examples to illustrate the relevance of infinite dimensionality to decision making under risk and under uncertainty.

In more detail, let X be a nonempty convex set, and $\succsim$ a preorder (a reflexive, transitive binary relation) on X. Consider the following axioms. The first is the standard strong independence axiom.

(SI) For $x, y, z \in X$ and $\alpha \in (0, 1)$,

$$x \succsim y \iff \alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z.$$

Thus $\succsim$ is an 'SI preorder'. We will be considering the following three Archimedean or continuity axioms.

(Ar) For $x, y, z \in X$, if $x \succ y \succ z$, then $(1-\epsilon)x + \epsilon z \succ y$ for some $\epsilon \in (0, 1)$.

(Ar$^+$) For $x, y, z \in X$, if $x \succ y$, then $(1-\epsilon)x + \epsilon z \succ y$ for some $\epsilon \in (0, 1)$.

(MC) For $x, y, z \in X$, if $\epsilon x + (1-\epsilon)y \succ z$ for all $\epsilon \in (0, 1]$, then $y \succsim z$.

The axiom Ar is weaker than, but for SI preorders equivalent to, the standard Archimedean axiom introduced by Blackwell and Girshick (1954).[1] It is weaker than the axiom Ar$^+$, essentially

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[1] See Proposition 5.

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introduced by Aumann (1962).[2] But both Ar and Ar^+ express a similar heuristic. Suppose x is strictly preferred to y, and z is some third alternative. Then Ar says that z cannot be so radically worse than y that a sufficiently small chance of z would disturb the original preference. The axiom Ar^+ extends this by replacing 'worse than' with 'worse than or incomparable with'. For SI preorders, MC is equivalent to the 'mixture-continuity' axiom of Herstein and Milnor (1953), that $\{\alpha \in [0,1] : \alpha x + (1-\alpha)y \succsim z\}$ is closed in $[0, 1]$. The displayed formulation is especially normatively natural, and suggests that MC should be seen as just as much an Archimedean condition as Ar and Ar^+.

The final two axioms restrict the possibilities for incomparability. Say that two members x and y of X are *comparable* if $x \succsim y$ or $y \succsim x$. They are *incomparable* if they are not comparable. They have a common upper bound if $z \succsim x$ and $z \succsim y$ for some $z \in X$, and similarly for common lower bound. Recall that an equivalence relation is a binary relation that is reflexive, symmetric, and transitive. The next axiom is nonstandard, while the last is the standard completeness axiom.

(Eq) Comparability is an equivalence relation.

(C) All members of X are comparable.

Using transitivity of $\succsim$, Eq is easily seen to be equivalent to the less technically convenient, but more illuminating

(**Eq**$'$) For all $x, y \in X$, if x and y are $\succsim$-incomparable, they have neither a common $\succsim$-upper bound, nor a common $\succsim$-lower bound.

This condition does not rule out incomparability. However, in realistic cases where an agent finds it hard to compare two alternatives, she will find it easy to imagine either an alternative that she finds superior to both, or an alternative she finds inferior to both, in each case violating Eq$'$. Thus while C excludes all incomparability, Eq excludes it in most cases of practical interest.

To state our main result, say that $\succsim$ is *nontrivial* if it has a nonempty strict part; that is, for some $x, y \in X$, $x \succ y$.

Theorem 1. *For any SI preorder $\succsim$ on a convex set X:*
(1) Any two of the following imply the third: MC, Ar and Eq.
(2) For nontrivial $\succsim$, any two of the following imply the third: MC, Ar^+, and C.

The following examples show that various strengthenings of this result are unavailable.

Example 2. (a) Let $X = \mathbb{R}^2$. Set $x \succsim y \Leftrightarrow x_1 \geq y_1$ and $x_2 = y_2$. Then $\succsim$ is nontrivial, MC, Ar, and Eq hold, but Ar^+ and C fail. Thus in (1), Ar cannot be replaced by Ar^+, and Eq cannot be replaced by C.

(b) Let X contain at least two elements. Set $x \succsim y \Leftrightarrow x = y$. Then $\succsim$ is trivial, MC and Ar^+ hold, but C fails. Thus in (2), 'nontrivial' cannot be dropped, and by (a), Ar^+ cannot be replaced by Ar.

Theorem 1 has several precedents. In a seemingly overlooked observation, Aumann (1962) claimed without proof that MC and Ar^+ imply Eq. Thus both parts of the theorem strengthen his claim. In Corollary 11 we strengthen another of Aumann's claims concerning the special case in which X is a vector space.

In the case where X is the set of probability functions on a given finite set, and thus can be identified with the standard simplex of a finite dimensional vector space, the second part of Theorem 1 was proved by Dubra (2011), building on Schmeidler (1971). Dubra's proof makes essential use of finite dimensionality. But placing no restrictions on the dimension of X allows for considerably broader applications,[3] including general sets of probability measures.

Schmeidler's result was that if a nontrivial preorder on a connected topological set has closed weak upper and lower contour sets, and open strict upper and lower contour sets, it must be complete. The axioms we discuss are purely algebraic, making them applicable to cases in which X is not naturally equipped with a topology. Our proof of Theorem 1 is purely algebraic. Indeed the main technical tool, stated in Theorem 6, states equivalences between the three continuity conditions and conditions involving algebraic openness or closedness in ambient vector spaces.

1.1. Discussion

The abstract structure of incomplete SI preorders on convex sets has been discussed, but the relevance of Theorem 1 perhaps has more to do with its compatibility with the typical concrete settings that are used to represent objective risk and subjective uncertainty. To illustrate, let Y be an arbitrary set, Y_c be a compact metric space, and Y_m an arbitrary measurable space; these are typical consequence spaces. Let $P(Y)$ be the set of finitely supported probability measures on Y, $P(Y_c)$ be the set of Borel probability measures on Y_c, and $P(Y_m)$ be an arbitrary convex set of probability measures on Y_m. These are obviously all convex sets, and cover typical cases involving objective risk. Let S_0 and S be a finite and arbitrary sets of states of nature respectively. Then $P(Y)^{S_0}$ is the set of Anscombe–Aumann 'horse lotteries'. Here, members of S_0 are bearers of subjective uncertainty, while the outcomes of horse lotteries are 'roulette lotteries' involving objective risk. For any x, $y \in P(Y)^{S_0}$ and $\alpha \in [0, 1]$, $\alpha x + (1-\alpha)y \in P(Y)^{S_0}$ is defined by setting $(\alpha x + (1-\alpha)y)(s) = \alpha x(s) + (1-\alpha)y(s)$ for any $s \in S_0$, making $P(Y)^{S_0}$ a convex set. Finally, the set Y^S is the set of Savage-acts associating states of nature with consequences; states of nature continue to be the bearers of subjective uncertainty, but no objective risk is modeled. The space Y^S is not naturally a convex set, but given a preorder on Y^S that satisfies reasonably modest axioms, Y^S can be endowed with convex structure; see for example Ghirardato et al. (2003). The importance of allowing X to be infinite dimensional can be seen from the fact that none of these typical domains can be identified with a finite dimensional X. There are many works discussing incomplete SI preorders in the settings just mentioned. Some focus only on SI strict partial orders,[4] but our results are still relevant, as every SI strict partial order is the asymmetric part of some SI preorder.

Given Theorem 6, it is natural to think of Ar and Ar^+ as 'open' conditions, and MC as a 'closed' condition. Both styles of condition have been used extensively in discussions of incomplete SI preorders on the kinds of convex sets just described. In almost every case we know of,[5] the open conditions are at least as strong as Ar in the given model, and the closed conditions are at least as strong as, and typically much stronger than, MC.[6] Thus Theorem 1 has considerable relevance.

2 The axiom Aumann actually discusses is $\epsilon_0 x + (1-\epsilon_0)z \succ y \Rightarrow \epsilon x + (1-\epsilon)z \succ y$ for all ϵ close enough to ϵ_0, but for SI preorders, this is equivalent to Ar^+.

3 Recall that the dimension of a convex set X is the dimension of $\mathrm{Span}(X - X)$, or, equivalently, the dimension of the smallest affine space containing X.

4 A strict partial order is a binary relation that is transitive, irreflexive, and asymmetric.

5 The exceptions are Aumann (1962), who imposes a continuity condition that is strictly weaker than both Ar and MC, and Seidenfeld et al. (1995) who impose a similar condition in the Anscombe–Aumann setting.

6 For open conditions, see Bewley (2002), Manzini and Mariotti (2008), Galaabaatar and Karni (2012, 2013), Evren (2014) and McCarthy et al. (2017c). For closed conditions, see Shapley and Baucells (1998), Ghirardato et al. (2003), Dubra et al. (2004), Nau (2006), Baucells and Shapley (2008), Evren (2008), Kopylov (2009), Gilboa et al. (2010), Danan et al. (2012), Ok et al. (2012) and McCarthy et al. (2017a). Without any continuity condition, one faces incomplete analogues of the situation analyzed by Hausner and Wendel (1952); see Borie (2016), Hara et al. (2016) and McCarthy et al. (2017b).