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Risk apportionment and multiply monotone targets

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HIGHLIGHTS

- A new interpretation of risk apportionment in the target-oriented decision-making model is proposed.
- These risk attitudes translate into smaller targets, expressing the decision maker's conservative behavior.
- Risk apportionment also means that the decision makers consider expected shortfalls with respect to an auxiliary target.

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ABSTRACT

This short note proposes a new interpretation of risk apportionment, in the target-oriented decision-making model. It is shown that these risk attitudes translate into smaller targets, expressing the decision maker's conservative behavior.

1. Introduction

Assume that decision makers act as if they maximize their respective expected utility of wealth. Specifically, a decision maker with utility function u prefers the random end-of-period wealth Y over X if $\mathrm{E}[u(X)] \leq \mathrm{E}[u(Y)]$, that is, if the expected utility for Y exceeds that for X.

In the expected utility setting, many risk attitudes are defined by properties of successive derivatives $u^{(1)}, u^{(2)}, u^{(3)}, \ldots$ of the utility function u. According to Eeckhoudt and Schlesinger (2006), preferences are said to satisfy risk apportionment of degree k if

$$(-1)^{k+1}u^{(k)} \geq 0$$

for some positive integer k. This risk attitude is justified by means of comparison of specific lotteries and is shown to correspond to the preference for "pain disaggregation", i.e. the decision maker prefers to spread the pains among different states of the world instead of concentrating them in a single state. Later on, Eeckhoudt et al. (2009) justified risk apportionment based on the tendency to combine "good with bad", where "good" and "bad" are defined by means of stochastic dominance rules.

For low degrees, risk apportionment corresponds to classical risk attitudes. Recall that in the expected utility model risk aversion, prudence, temperance, and edginess are defined respectively by a negative second derivative, by a positive third derivative, by a negative fourth derivative, and by a positive fifth derivative of the utility function. Thus, these notions appear to be particular cases of risk apportionment of degrees 2 to 5. Prudence is justified after Kimball (1990) by reference to the decision of building up precautionary savings in order to better face future income risk. Kimball (1992) and Gollier and Pratt (1996) justified temperance based on the demand for risky assets in the presence of background risks, when an unavoidable background risk leads the decision maker to reduce exposure to another risk even if the two risks are statistically independent. Edginess was defined by Lajeri-Chaherli (2004) in a context of multiple risks in a two-period model, capturing the reactivity to multiple risks on precautionary motives. In Denuit and Rey (2010), the concept of risk apportionment is linked to the tendency of decision makers to become less sensitive to an increase in correlation when richer. See also Denuit and Rey (2013).

In this paper, we propose a new interpretation of these concepts in the target-based counterpart to the expected utility model. The remainder of this paper is organized as follows. Section 2 recalls the representation of expected utility by means of target functions. Section 3 relates risk apportionment to a property of the probability density function of the target, known as multiple monotonicity. The final Section 4 briefly concludes the paper. The new results and their interpretation are summarized there.

2. Target-oriented decision-making model

Consider an economic agent faced with two strategies, leading to end-of-period wealth levels X and Y. To decide which one is the most attractive, the agent could first select a target and then ranks X and Y according to the probability that they reach the specified

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target. Such a decision maker acts in order to maximize the probability of meeting the target. Despite its apparent difference with the expected utility paradigm, it turns out that this procedure is closely related to expected utility maximization, as explained next.

Let us restrict our analysis to bounded, continuous utility functions defined on $\mathbb{R}^+ = [0, +\infty)$. Following Borch (1968) and Berhold (1973), let us normalize the utility function u so that it can be interpreted as the distribution function of a random target T to reach. By convention, $u(0) = 0$ so that only positive targets T are considered. We thus have

$$u(x) = \mathrm{P}[T \leq x].$$

Assuming that the random wealth X and the target T are mutually independent and denoting as F_X the distribution function of X, we have

$$\begin{aligned} \mathrm{E}[u(X)] &= \int_0^\infty u(x)dF_X(x) \\ &= \int_0^\infty \mathrm{P}[T \leq x]dF_X(x) \\ &= \mathrm{P}[X \geq T]. \end{aligned} \tag{2.1}$$

As pointed out by Castagnoli and LiCalzi (1996) and Bordley and LiCalzi (2000), (2.1) shows that maximizing the expected utility of X is equivalent to maximizing the probability that X meets the random target T corresponding to the normalized utility function u. The target-based approach is phrased entirely in the language of probability, making it simpler to explain and to use.

The "size" of the target reflects the risk attitude. A decision maker with a "smaller" target is easier to satisfy as the target to reach is comparatively smaller, whereas a decision maker with a "larger" target is likely to be more willing to take risk. Hence, risk attitudes can be associated to the respective size of the targets.

In this paper, we aim to obtain new interpretations for the concept of risk apportionment in terms of targets. The following example illustrates the approach followed in the present work. Consider for instance risk aversion, or risk apportionment of degree 2. A risk-averse decision maker always prefers a certain yield $\mathrm{E}[X]$ to the risky wealth X, whatever the distribution of X. The utility function then satisfies $\mathrm{E}[u(X)] \leq u(\mathrm{E}[X])$ for all X. It is a simple consequence of Jensen's inequality that this holds if, and only if, u is concave. As u is the distribution function of the target, u'' is the derivative of the probability density function of the target, and risk aversion $\Leftrightarrow u'' \leq 0$ thus means that T has a decreasing probability density function (as pointed out in Section 3 of Castagnoli and LiCalzi, 1996). A target with decreasing probability density expresses conservatism in decision making as the most likely targets to meet are the worst outcomes. At degree 2, risk apportionment is thus associated to targets with decreasing probability density function. In the remainder of this paper, we extend this result to higher degrees $s = 3, 4, \ldots,$ replacing decreasing probability density functions with multiply monotone ones.

3. Multiply monotone targets

3.1. Risk apportionment and multiply monotone probability density functions

Completely monotone functions are non-increasing functions with derivatives of all degrees alternating in signs. Such functions correspond to scale mixtures of negative exponentials. Functions on $\mathbb{R}^+$ satisfying this property up to some degree $k < \infty$ have been studied in applied probability, under the name k-monotone.

In this paper, we consider targets with $(s-1)$-monotone probability density function, i.e. a decreasing density over $\mathbb{R}^+$, with derivatives of degrees 1 to $s-1$ alternating in signs. When $s = 2$, we recover targets with decreasing probability density functions that are known to be associated to risk aversion.

Targets obeying the exponential distribution, i.e. $\mathrm{P}[T \leq t] = 1 - \exp(-ct)$, $t \geq 0$, for some $c > 0$, fulfill this condition for all s. They correspond to the exponential utilities expressing constant absolute risk aversion. All probabilistic mixtures of exponential distributions, obtained by letting the parameter c become random, also satisfy this requirement. They correspond to mixed risk aversion in the sense of Caballe and Pomansky (1996). Power laws of the form $\mathrm{P}[T \leq t] = 1-(1+t)^{-c}$, $t \geq 0$, with $c > 0$ also fulfill the aforementioned condition. See also Section 3.2 for the general form of targets with $(s-1)$-monotone probability density functions.

Let us now state the following elementary result, which related risk apportionment of degree s to targets with $(s-1)$-monotone probability density functions. The proof simply follows from the fact that the kth derivative $u^{(k)}$ of the utility function is the $(k-1)$th derivative of the probability density function of the associated target.

Property 3.1. *In the target-based approach to decision making, preferences exhibiting risk apportionment of degrees $k = 1$ to $s \in \{2, 3, 4, \ldots\}$ correspond to targets with $(s-1)$-monotone probability density function, i.e. a decreasing density over $\mathbb{R}^+$, with derivatives of degrees 1 to $s-1$ alternating in signs.*

This means that a risk-averse decision maker ($s = 2$) possesses a target with a decreasing density over $\mathbb{R}^+$. Prudent, risk-averse decision makers ($s = 3$) then have a target with a decreasing and convex probability density function. And so on.

In the next section, based on a representation theorem for random variables with multiply monotone probability density function, we show that the corresponding targets become smaller when s increases, providing a new interpretation of risk apportionment.

3.2. Risk apportionment and scaled minima of uniform random variables

Khinchine representation theorem (see, e.g., Theorem 1.3 in Dharmadhikari and Joag-Dev, 1988) ensures that a target T has a decreasing probability density function over $\mathbb{R}^+$ if, and only if, there exist independent random variables U and Z, such that U is uniformly distributed on [0, 1] and the product UZ is distributed as T. Hence, every risk-averse decision maker possesses a target of the form UZ. This result extends to higher degrees and provides us with the following representation of targets possessing multiply monotone probability density functions.

Proposition 3.2. *Consider a sequence of independent random variables $U_1, U_2, \ldots,$ uniformly distributed over the unit interval and $s \in \{2, 3, 4, \ldots\}$. In the target-based approach to decision making, preferences exhibit risk apportionment of degrees 1 to s if, and only if, the target T is distributed as $Z \min\{U_1, \ldots, U_{s-1}\}$ for some positive random variable Z independent of $U_1, \ldots, U_{s-1}$.*

Proof. Recall from Lefevre and Loisel (2013, Proposition 2.1) that the target T possesses a $(s-1)$-monotone probability density function if, and only if, it is distributed as $(1 - U^{\frac{1}{s-1}})Z$ where U is uniformly distributed over [0, 1] and Z is valued in $\mathbb{R}^+$ and independent of U. Such targets are known to correspond to preferences exhibiting risk apportionment of degrees 1 to s according to Property 3.1. Denoting as $y_+ = \max\{y, 0\}$ the positive part of y, equal to y if $y > 0$ and to 0 otherwise,

$$\begin{aligned} \mathrm{P}[T \leq t] &= \mathrm{P}\left[\left(1 - U^{\frac{1}{s-1}}\right) Z \leq t\right] \\ &= \mathrm{P}\left[U \geq \left(1 - \frac{t}{Z}\right)^{s-1}\right] \end{aligned}$$

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