ARTICLE IN PRESS
Mathematical Social Sciences ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

On characterizations of the probabilistic serial mechanism involving incentive and invariance properties☆

Onur Kesten[a,*], Morimitsu Kurino[b], M. Utku Ünver[c]

[a] Carnegie Mellon University, Tepper Business School, Pittsburgh, PA, 15213, USA
[b] University of Tsukuba, Faculty of Engineering, Information and Systems, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8573, Japan
[c] Boston College, Department of Economics, Chestnut Hill, MA, 02167, USA

HIGHLIGHTS

- Probabilistic serial is ordinal efficient, envy-free and weakly strategyproof.
- PS is *not* characterized by the three properties.
- Weak strategy proofness is logically independent of any invariance property.

ARTICLE INFO

Article history:
Received 24 February 2016
Received in revised form
14 November 2016
Accepted 28 November 2016
Available online xxxx

ABSTRACT

This paper studies the problem of assigning n indivisible objects to n agents when each agent consumes one object and monetary transfers are not allowed. Bogomolnaia and Moulin (2001) proved that for $n = 3$, the probabilistic serial mechanism is characterized by the three axioms of ordinal efficiency, envy-freeness, and weak strategy-proofness. We show that this characterization does not extend to problems of arbitrary size; in particular, it does not hold for any $n \geq 5$. A number of general characterizations of the probabilistic serial mechanism have been obtained in the recent literature by replacing weak strategy-proofness with various invariance axioms while retaining ordinal efficiency and envy-freeness. We show that weak strategy-proofness is in fact logically independent of all invariance axioms used in these characterizations.

1. Introduction

Many real-life problems such as school choice, organ transplantation, and on-campus housing involve the assignment of discrete indivisible objects without the use of monetary transfers. We consider the simplest discrete resource allocation problem in which n objects are assigned to n agents who have strict preferences over objects. A *mechanism* is a rule that specifies a stochastic assignment of objects to agents based on their reported preferences. The widely-used mechanism for this type of problems in practice is the *random serial dictatorship* (RSD) mechanism: randomly order the agents and let them sequentially choose their favorite objects. RSD is well-known for its strategy-proofness and ex-post efficiency. In their seminal paper, Bogomolnaia and Moulin (2001) (BM hereafter) showed that RSD is neither ordinally efficient nor envy-free, but is weakly envy-free.

BM introduced the *probabilistic serial* (PS) mechanism as a major competitor to RSD. The outcome of PS is computed via the *simultaneous eating algorithm* (SEA): Imagine that each object is a continuum of probability shares. Let agents simultaneously "eat away" from their favorite objects at the same speed; once the favorite object of an agent is gone, she turns to her next favorite object, and so on. We interpret the share of an object eaten away by an agent throughout the process as the probability PS assigns her that object.

PS is ordinally efficient and envy-free. This surprising observation in turn led to much attention being devoted to PS and its

☆ We would like to thank three anonymous referees and participants at Duke "Roth–Sotomayor: 20 Years After" Conference, CORE, Kyoto, Maastricht, Osaka, and Tsukuba for comments. Kurino acknowledges the financial support from JSPS KAKENHI Grant number 15K13002. Ünver acknowledges the research support of Microsoft Research Lab, New England.

* Corresponding author.
E-mail addresses: okesten@andrew.cmu.edu (O. Kesten), kurino@sk.tsukuba.ac.jp (M. Kurino), unver@bc.edu (M.U. Ünver).

http://dx.doi.org/10.1016/j.mathsocsci.2016.11.005
0165-4896/

Please cite this article in press as: Kesten, O., et al., On characterizations of the probabilistic serial mechanism involving incentive and invariance properties. Mathematical Social Sciences (2016), http://dx.doi.org/10.1016/j.mathsocsci.2016.11.005

various extensions[1] and characterizations. Unlike RSD which is strategy-proof, PS is weakly strategy-proof. BM provided a first characterization of PS through ordinal efficiency, envy-freeness, and weak strategy-proofness with the added condition that there are three agents. A generalization of the original BM characterization to an arbitrary number of agents/objects has thus far been elusive. We specifically ask whether the BM characterization result holds for problems of arbitrary size and give a negative answer to this question. In particular, we construct a highly non-trivial mechanism for the case of five agents, different from PS, which satisfies ordinal efficiency, envy-freeness, and weak strategy-proofness (Lemma 1). Building on this construction, we show that, when there are at least five agents, it is possible to obtain a mechanism different from PS, which satisfies the three properties (Proposition 1).

Recently several papers provide various PS characterizations for more general settings with arbitrary number of agents and possibly for multiple copies of objects. A common theme in these characterizations is the use of ordinal efficiency and envy-freeness along with an invariance/monotonicity type property that requires the robustness of the random assignment to certain perturbations of agents' preferences.[2] In light of Proposition 1, it may be tempting to think that the invariance properties used in these characterizations are stronger than weak strategy-proofness. We show that no such conclusion is true. Indeed, we show that there is no logical connection between weak strategy-proofness and any of these invariance properties (Proposition 2).

Section 2 describes the formal model and Section 3 provides the main result. Section 4 concludes.

2. Model

A discrete resource allocation problem (cf. Hylland and Zeckhauser, 1979; Shapley and Scarf, 1974), or simply a **problem**, is a list $(N, A, \succ)$ where $N = \{1, \ldots, n\}$ is a finite set of agents; A is a finite set of objects with $|A| = |N| = n$; and $\succ = (\succ_i)_{i\in N}$ is a preference profile where $\succ_i$ is the strict preference relation of agent i on A. Let $\mathbf{P}$ be the set of preferences for any agent. Let $\succeq_i$ denote the weak preference relation induced by $\succ_i$. We assume that preferences are linear orders on A, i.e., for all $a, b \in A$ and all $i \in N$, $a \succeq_i b \Leftrightarrow a = b$ or $a \succ_i b$. We sometimes represent $\succ_i$ as an ordered list beginning with the most preferred object of agent i and continuing to her least. For example, given $A = \{a, b, c\}$, we interpret $\succ_i = (b, c, a)$ as $b \succ_i c \succ_i a$. A centralized authority shall assign objects to agents such that each agent receives exactly one object.

A **random allocation** for agent i is a vector $P_i = (p_{i,a})_{a\in A}$ where $p_{i,a} \in [0, 1]$ denotes the probability that agent i receives object a, and $\sum_{a\in A} p_{i,a} = 1$. A **random assignment**, denoted as $P = [P_i]_{i\in N} = [p_{i,a}]_{i\in N, a\in A}$, is a bistochastic matrix, i.e., $\sum_{a\in A} p_{i,a} = 1$ for all $i \in N$ and $\sum_{i\in N} p_{i,a} = 1$ for all $a \in A$. Let $\mathcal{R}$ be the set of random assignments.

Observe that a random assignment gives only the marginal probability distribution according to which each agent will be assigned an object. It does not specify the distribution according to which objects should jointly be assigned to agents. To define this joint probability distribution, we first need to define (deterministic) assignments and probability distributions over them. An **assignment** is an element $P \in \mathcal{R}$ such that $p_{i,a} \in \{0, 1\}$ for all $i \in N$ and all $a \in A$. Let $\mathcal{A}$ be the set of assignments. A **lottery** $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}}$ is a probability distribution over assignments, i.e., $\lambda_\alpha \in [0, 1]$ for all $\alpha \in \mathcal{A}$ and $\sum_{\alpha\in\mathcal{A}} \lambda_\alpha = 1$.

Clearly, each lottery induces a random assignment. Let $P^\lambda \in \mathcal{R}$ be the random assignment induced by lottery λ, i.e., $p^\lambda_{i,a} = \sum_{\alpha\in\mathcal{A}:\alpha_{i,a}=1} \lambda_\alpha$ for all $i \in N$ and all $a \in A$. It turns out that the converse statement is also true: For each $P \in \mathcal{R}$ there exists a lottery λ that induces it, i.e., $P^\lambda = P$ (Birkhoff, 1946; von Neumann, 1953). Thus, the centralized authority can simply restrict attention to random assignments rather than lotteries.[3]

Let $i \in N$, $a \in A$, $\succ_i \in \mathbf{P}$, and $P, R \in \mathcal{R}$ be given. Let $U(\succ_i, a) = \{b \in A \mid b \succeq_i a\}$ be the upper contour set of object a under at $\succ_i$. Let $F(\succ_i, a, P_i) = \sum_{b\in U(\succ_i,a)} p_{i,b}$ be the probability that i is assigned an object at least as good as a under P_i. Moreover, P_i **stochastically dominates** R_i at $\succ_i$ if $F(\succ_i, a, P_i) \geq F(\succ_i, a, R_i)$ for all $a \in A$. In addition, P **stochastically dominates** R at $\succ$ if P_i stochastically dominates R_i at $\succ_i$ for all $i \in N$.

We are now ready to introduce a powerful efficiency notion. A random assignment is **ordinally efficient** at $\succ$ if it is not stochastically dominated by another random assignment at $\succ$. BM characterized ordinal efficiency by acyclicity as follows. A random assignment P is ordinally efficient at $\succ$ if and only if there is no cycle $(a_1, i_1, a_2, i_2, \ldots, a_m, i_m, a_{m+1})$ such that $a_1 = a_{m+1}$, and for each $\ell \in \{1, \ldots, m\}$, $a_\ell \succ_{i_\ell} a_{\ell+1}$ and $p_{i_\ell,a_{\ell+1}} > 0$.

Our fairness property is a fundamental principle in mechanism design theory originally proposed by Foley (1967). A random assignment is envy-free if each agent, regardless of her vNM utilities, weakly prefers her random allocation to that of any other agent. Formally, given $\succ \in \mathbf{P}^N$, $P \in \mathcal{R}$ is **envy-free** at $\succ$ if for all $i \in N$, P_i stochastically dominates P_j for all $j \in N$ at $\succ_i$.

A mechanism is a systematic way of finding a random assignment for a given problem. Formally, a **mechanism** is an allocation rule $\varphi : \mathbf{P}^N \to \mathcal{R}$. A mechanism is said to satisfy a property if its outcome, for any problem, satisfies that property. A mechanism φ is **weakly strategy-proof** if no agent ever stochastically gains by misreporting her preferences, i.e., for all $\succ$ and all $i \in N$, there is no $\succ'_i$ such that $\varphi_i(\succ'_i, \succ_{-i})$ stochastically dominates $\varphi_i(\succ_i, \succ_{-i})$ at $\succ_i$ and $\varphi_i(\succ'_i, \succ_{-i}) \neq \varphi_i(\succ_i, \succ_{-i})$.

BM introduced the **probabilistic serial mechanism** (**PS**),[4] the outcome of which can be computed via the following simultaneous eating algorithm (SEA):

Given a problem $\succ$, think of each object a as an infinitely divisible good with supply of 1.

Step 1: Each agent "eats away" from her favorite object at the same unit speed. Proceed to the next step when an object is completely exhausted.

⋮

Step s, for $s \in \{2, \ldots, S\}$: Each agent eats away from her remaining favorite object at the same unit speed. Proceed to the next step when an object is completely exhausted.

1 See, for example, Kojima (2009), Yılmaz (2010), Athanassoglou and Sethuraman (2011).

2 Kesten et al. (2011) is the first paper of this kind that uses an upper invariance property, which was later refined by Bogomolnaia and Heo (2012). Hashimoto and Hirata (2011) characterizes PS via truncation robustness in a model, different from ours, which assumes outside options. The weakest property among these auxiliary properties, which characterizes PS along with ordinal efficiency and envy-freeness is the weak invariance property by Hashimoto et al. (2014).

3 Once a random assignment is determined, finding a lottery that induces it is computationally easy. Birkhoff–von Neumann Theorem's proof is constructive (cf. von Neumann, 1953); each assignment in the support of the lottery can be computed in polynomial time using the well-known Hungarian algorithm (Kuhn, 1955). Moreover, there are at most n^2 steps in the construction and such assignments in the support. Therefore, the lottery can be computed in polynomial time. Also see Hylland and Zeckhauser (1979)'s appendix for an alternative construction.

4 PS was initially proposed by Crès and Moulin (2001) for a model where agents have the same rankings over objects.