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On construction of social welfare orders satisfying Hammond equity and Weak Pareto axioms☆

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HIGHLIGHTS

- Characterize domain Y for constructive social welfare order satisfying HE and WP.
- If $(Y, <)$ is not a well-ordered set, the SWO satisfying HE and WP is non-constructive.
- Show the correspondence principle holds in this situation.

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ABSTRACT

This paper examines the constructive nature of a social welfare order that respects Hammond equity axiom and Weak Pareto axiom. It describes the domains (of the one period utilities) on which an explicit construction is possible. A social welfare order satisfying the Hammond equity and Weak Pareto admits an explicit construction if and only if the domain is a well-ordered set.

1. Introduction

Intergenerational equity principles have received wide attention in the recent social choice literature.[1] The literature deals with questions like aggregation of utilities/welfares of an infinite number of generations or agents by either real valued function (social welfare function) or complete and transitive binary relations (social welfare order) which are equitable (i.e., satisfying some principles of equity and efficiency). While there are many variants of equity principles, they could be divided in two broad categories, namely, procedural and consequentialist. For example, the *Anonymity* principle is a procedural equity notion, *Pigou–Dalton* equity, *Strong* equity and *Hammond* equity are some of the important consequentialist equity principles.

We consider binary relations on the set of infinite utility streams, $X = [0, 1]^{\mathbb{N}}$ satisfying Hammond equity. Hammond equity deals with situations in which the *distribution* of utilities of generations changes in a particular manner. It is one of the key consequentialist equity concepts, the other being the Pigou–Dalton transfer principle.[2] It was introduced by Hammond (1976), who called it the *Equity* Axiom, and is in the spirit of the *Weak Equity* Axiom of Sen (1973).

There are several representation results, some positive but mostly negative, for equitable infinite utility streams, starting with Diamond (1965) for anonymity when combined with Strong Pareto. The case of Hammond equity is similar. On the one hand, combining it with Strong Pareto leads to no representation even when it is barely possible to compare different infinite utility streams in a non-trivial manner, on the other, weakening the efficiency criteria to Weak Pareto leads to existence of real valued social welfare function for richer sets of infinite utility streams [see Alcantud and Garcia-Sanz, 2013]. Dubey and Mitra (2015) characterized the subsets Y of $[0, 1]$ for which social welfare functions satisfying both the axioms exist. The social welfare functions exist if and only if $(Y, <)$ is a *well-ordered* set, where the inequality $<$ is inherited from the ordered set $(\mathbb{R}, <)$.

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1 The classic contributions on the subject are Koopmans (1960), Koopmans et al. (1964) and Diamond (1965).

2 Hammond equity has several variations which have been discussed in the literature. Strong Equity (see d'Aspremont and Gevers, 1977, and Dubey and Mitra, 2014a) and Hammond equity for the Future (see Asheim et al., 2007 and Banerjee, 2006) are notable variations. The Pigou–Dalton transfer principle has been discussed in Hara et al. (2008); Altruistic Equity, a variation of the Pigou–Dalton transfer principle, has been discussed by Sakamoto (2012).

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The lack of representation led economists to explore if it is possible to specify any social welfare order on the set of infinite utility streams. This line of research led to mostly positive outcomes, i.e., there exist social welfare orders satisfying equity and Strong Pareto. It opened the possibility of applying these social welfare orders in policy making provided they can be explicitly written down. Careful reading of these results reveal that the proofs invariably rely on some variant of the *Axiom of Choice* (AC). This led to emergence of the question, "Is AC necessary to establish the existence of social welfare orders?". An affirmative answer would imply that the social welfare order would be essentially unusable for any policy purposes as it would be a non-constructive object. It is noteworthy that a non-constructive proof establishes the existence of some mathematical object without providing any means of describing it.

Using a variation, due to Arrow (1951, p. 64), of Szpilrajn's Lemma (Szpilrajn, 1930) Svensson (1980) showed that a complete ordering of infinite utility streams satisfying Strong Pareto and anonymity exists. Szpilrajn's Lemma is usually established using Zorn's Lemma, which is equivalent to AC, hence it is a non-constructive technique. Thus the issue of necessity of using some non-constructive device, in the proof of Svensson (1980) result invited further scrutiny. Fleurbaey and Michel (2003) investigated this question in detail and conjectured "There exists no *explicit* (that is, avoiding the axiom of choice or similar contrivances) description of an ordering which satisfies weak Pareto and indifference to finite permutations."

Zame (2007) and Lauwers (2010) were the first two to confirm this conjecture using two different techniques. Zame (2007) proved that the AC is necessary to prove existence of social welfare orders as it entails the existence of a non-measurable set which is a non-constructive object. Lauwers (2010) showed similar result by showing existence of non-Ramsey set which is again a non-constructive device. The technique devised by Lauwers (2010) turned out to be quite versatile and has been used to examine the role of AC in existence of efficient social welfare orders respecting various equity principles (both procedural and consequentialist).[3] We use Lauwers' approach (described in Lauwers, 2010, and Lauwers, 2012) and show that if a social welfare order satisfying Hammond equity and Weak Pareto exists for domain $Y \subset [0, 1]$, which is not well-ordered, then such an order is necessarily *non-constructive*.

Similar lines of inquiry adopted in Dubey (2011), Dubey and Mitra (2014a), Dubey and Mitra (2015), and Dubey (2016) have led to emergence of a *correspondence* principle, namely, the sets Y for which there exists equitable and efficient social welfare function are precisely the same as the ones for which we also have social welfare orders admitting explicit construction.[4]

Our objective in this paper is to examine the *extent* to which Hammond equity comes into conflict with Weak Pareto, while insisting on explicit description of social welfare orders on the space of infinite utility streams. It is well known in the literature that such social welfare orders exist for general space of infinite utility streams, $X = Y^{\mathbb{N}}$ for every set $Y \subset [0, 1]$ (see Bossert et al., 2007), *unlike* the social welfare functions satisfying these properties, which exist only for $Y \subset [0, 1]$ for which $(Y, <)$ is a well-ordered set, (see Dubey and Mitra, 2015, Theorem 1), a fairly restricted class of subsets of $[0, 1]$.

Our choice of efficiency principle requires some explanation. It is known (see Alcantud and Garcia-Sanz, 2013) that there is no social welfare function satisfying Hammond equity and Strong Pareto, if the domain Y contains at least four distinct elements. That is, an impossibility result arises as soon as we admit a situation in which Hammond equity can play a role in ranking two utility streams. However, if we weaken the efficiency principle to Monotonicity, the combination of Hammond equity and Monotonicity would clearly be satisfied by the trivial social welfare function which assigns the same welfare number to *all* utility streams. We choose a middle ground and focus on the efficiency principle of Weak Pareto, having more bite than Monotonicity and certainly weaker than the Strong Pareto.

The goal in this paper, then, is to completely characterize the domain $Y \subset [0, 1]$ for which the social welfare orders on the space of utility streams $X = Y^{\mathbb{N}}$, satisfying Hammond equity and Weak Pareto admit explicit construction. The principle result of this paper (Theorem 1) shows that social welfare orders satisfying Hammond equity and Weak Pareto are constructive if and only if the domain Y is a well-ordered set. In other words, the domain Y for which the social welfare order is non-constructive is precisely the same as the one for which no social welfare function exists. Thus a correspondence principle holds true in this scenario, much like in other similar situations.

Rest of the paper is organized as follows. Section 2 contains all the definitions and a brief description of well-ordered sets. In Section 3, we state and prove the main result of the paper (Theorem 1). The conclusions are in Section 4.

2. Preliminaries

2.1. Notation

Let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{M}$ be the sets of real numbers, natural numbers, and negative integers respectively. For all $y, z \in \mathbb{R}^{\mathbb{N}}$, we write $y \geq z$ if $y_n \geq z_n$, for all $n \in \mathbb{N}$; we write $y > z$ if $y \geq z$ and $y \neq z$; and we write $y \gg z$ if $y_n > z_n$ for all $n \in \mathbb{N}$.

2.2. Strictly ordered sets, order types and well-ordered sets

We present a brief description of some of the concepts from the mathematics literature dealing with *strictly ordered* sets, *order types* and well-ordered sets. The exposition here broadly follows Dubey and Mitra (2015, sub-section 2.2). Set A is said to be strictly ordered by a binary relation $\Re$ if $\Re$ is *connected*,

if $a, a' \in A$ and $a \neq a'$, then either $a \Re a'$ or $a' \Re a$ holds,

transitive,

if $a, a', a'' \in A$ and $a \Re a'$ and $a' \Re a''$ hold, then $a \Re a''$ holds,

and *irreflexive*,

$a \Re a$ holds for no $a \in A$.

In this case, the strictly ordered set will be denoted by $(A, \Re)$. For example, the set $(\mathbb{N}, >)$ with $>$ the usual 'greater than' relation on $\mathbb{R}$ is strictly ordered.

We say that a strictly ordered set $(A', \Re')$ is *similar* to the strictly ordered set $(A, \Re)$ if there is a one-to-one function f mapping A

[3] Some of the recent papers using Lauwers (2010) approach are Dubey (2011), Banerjee and Dubey (2014), Dubey and Mitra (2014a), Dubey and Mitra (2015) and Dubey (2016). Although these papers deal with equity principles in general, Banerjee and Dubey (2014) is a notable exception. It examines *impatience* implications of constructive monotone social welfare orders.

[4] Dubey (2016) deals with Pigou–Dalton transfer principle. Social welfare orders satisfying Pigou–Dalton transfer principle and Weak Pareto axiom have been shown to exist, (see Bossert et al., 2007, Theorem 1). However, they cannot be represented as has been proved in Alcantud (2012, Proposition 1) and Sakamoto (2012, Proposition 5). Further, Sakamoto (2012) and Alcantud (2010) have shown, relying on AC, that social welfare function satisfying Pigou–Dalton transfer principle exists. In this context, Dubey (2016) shows that a modified form of correspondence principle for social welfare orders holds. Thus social welfare order satisfying Pigou–Dalton transfer principle is constructive if and only if the social welfare function admits construction.