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Returns to scale and the random production of innovations

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HIGHLIGHTS

- I define returns to scale in processes of research that generate random innovations.
- I determine the functional form of innovation processes under constant returns.
- I determine the functional forms under increasing and decreasing returns.
- Constant – returns processes that use only one factor form a one – parameter family.

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ABSTRACT

I define constant, increasing and decreasing returns to scale in the production of innovations that occur randomly with a probability that depends upon resources spent in research. I analyse the mathematical representations of random processes of innovation that exhibit constant, increasing or decreasing returns to scale in that sense and determine their respective functional forms. I also give two complementary conditions, which are respectively sufficient for increasing returns to scale, and decreasing returns. Finally, as a particular case, I show processes that use only one factor of innovation and satisfy constant returns form a one—parameter family.

1. Introduction

In various economic models, one assumes innovations occur randomly with a probability that depends upon the inputs of a research process. Instances of such processes of innovation, which I will call stochastic research technologies, can be found in the literature on patent races (Tirole, 1988), or in Schumpeterian growth theory (Aghion and Howitt, 2008). In those models, the properties of research technologies are assumed *a priori*.

By way of contrast, my aim in the present paper is to derive the properties of stochastic research technologies from economic fundamentals. It is also to determine explicitly the mathematical representation of those processes. That exercise has both theoretical and empirical purposes. On theoretical grounds, its results will determine which assumptions can be rationalized, and which cannot. On empirical grounds, they will determine the functional forms that can be used for estimation or calibration.

The argument proceeds as follows. In Section 2, I define stochastic research technologies formally. In Section 3, I discuss their economic fundamentals and I define constant, increasing and decreasing returns to scale in the random production of innovations. In Section 4, I determine in full generality the respective functional forms of stochastic research technologies that exhibit constant, increasing or decreasing returns to scale. I also give two complementary conditions, which are respectively sufficient for a process of innovation to exhibit increasing returns to scale, or decreasing returns. Finally, as a particular case, I show processes that use only one factor of innovation and satisfy constant returns form a one—parameter family.

2. Stochastic research technologies

In informal terms, a stochastic research technology is a process of innovation that associates some probability of discovery to each and every combination of inputs, or factors of innovation. It is natural to assume an increase in the quantity of any factor of innovation cannot reduce the probability of success.

In more formal terms, we will define an "innovation function" as the mathematical representation of a stochastic research technology. We will thus assume there are $n \geq 1$ factor(s) of innovation, where n is a natural number. The quantity x_i of factor i is an element of the set of non-negative real numbers $[0, \infty[$, for $i = 1, 2, \ldots, n$, so that the vector of quantities $\vec{x} = (x_1, x_2, \ldots, x_n)$ is an element of $[0, \infty[^n \subset \mathbb{R}^n$, the Cartesian product of $[0, \infty[$

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by itself n times. By definition, a Bernoulli trial is a probability experiment that has two possible outcomes: success or failure (Parzen, 1960). Since we can index a Bernoulli experiment by the probability $0 \le p \le 1$ of its success, we will identify the set of Bernoulli experiments with $[0, 1]$. Finally, we will recall the definitions of two types of "increasing functions". We will use the first immediately to define innovation functions; and the second later in our analysis.

Definition 1. A function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is *monotone increasing in all arguments* if, and only if, we have $f(\vec{x}) \ge f(\vec{y})$, when any two elements $\vec{x}$ and $\vec{y}$ of S are such that $x_i \ge y_i$, for $i = 1, 2, \ldots, n$.

Definition 2. A function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is *strictly increasing in all arguments* if, and only if, we have $f(\vec{x}) > f(\vec{y})$, whenever any two elements $\vec{x}$ and $\vec{y}$ of S are such that $x_i \ge y_i$, for $i = 1, 2, \ldots, n$, and there is at least one k such that $x_k > y_k$.

Definition 3. A function p from $[0, \infty[^n$ onto $[0, 1]$ is an *innovation function* if, and only if, it is continuous and monotone increasing in all arguments.

3. The fundamentals of innovation and returns to scale

An analysis of economic fundamentals will allow me to motivate the definitions of returns to scale I will give with respect to the random production of innovations. Indeed, there is an old argument in microeconomic literature, according to which production technologies exhibit at least constant returns to scale, when all factors are variable. It consists in observing that, if some combination (A, B) of inputs A and outputs B can be obtained with some process of production, the combination $(2A, 2B)$ can be obtained simply by duplicating the process of production (Henderson and Quandt, 1980).

With respect to stochastic research technologies, we can make a similar argument. Let $\vec{x}$ be an element of $[0, \infty[^n$. When all factors of innovation are variable and one spends resources $2\vec{x}$ on research, one can duplicate the Bernoulli trial that is set up when one spends resources $\vec{x}$. More generally, for $m = 1, 2, \ldots$, when all factors of innovation are variable and one allocates resources $m\vec{x}$ to research, one can set up m identical and independent Bernoulli trial(s) with resources $\vec{x}$. As the sum of a sequence of independent and identically distributed (i.i.d.) random variables that arise from Bernoulli experiments follows a binomial distribution (Parzen, 1960), the probability there will be at least one success is given by $1-\left(1-p\left(\vec{x}\right)\right)^m$. When more than one trial leads to a discovery, we naturally interpret the event as an instance of multiple occurrences of the same discovery.

Definition 4. We will say an innovation function $p : [0, \infty[^n \to [0, 1]$ exhibits *constant returns to scale* if, and only if, we have, for all $\vec{x}$ in $[0, \infty[^n$ and $m = 1, 2, \ldots$,

$$p\left(m\vec{x}\right) = 1 - \left(1 - p\left(\vec{x}\right)\right)^m . \tag{1}$$

The left-hand side of (1) is the probability $p\left(m\vec{x}\right)$ of a discovery, when one spends resources $m\vec{x}$ on one single research project. The right-hand side is the probability of discovery corresponding to m independent projects with investment $\vec{x}$. Eq. (1) thus says that a large research project is equivalent to many uncoordinated smaller research projects.

Constant returns to scale is a property that a research process may or may not exhibit.

Definition 5. We will say an innovation function $p : [0, \infty[^n \to [0, 1]$ exhibits *increasing returns to scale* if, and only if, we have, for all $\vec{x} \neq \vec{0}$ in $[0, \infty[^n$ and all natural numbers $m \ge 2$,

$$p\left(m\vec{x}\right) > 1 - \left(1 - p\left(\vec{x}\right)\right)^m . \tag{2}$$

Definition 6. We will say, finally, an innovation function $p : [0, \infty[^n \to [0, 1]$ exhibits *decreasing returns to scale* if, and only if, we have, for all $\vec{x} \neq \vec{0}$ in $[0, \infty[^n$ and all natural numbers $m \ge 2$,

$$p\left(m\vec{x}\right) < 1 - \left(1 - p\left(\vec{x}\right)\right)^m . \tag{3}$$

4. The propositions

We can solve Eqs. (1)–(3) in full generality and derive from their solution a complete characterization of random processes of innovation that exhibit constant, increasing or decreasing returns to scale. We will proceed in three steps. As a first step, we will verify that we need to consider only two classes of probability functions, when we discuss returns to scale: those for which $p\left(\vec{x}\right) \equiv 1$; and those for which $0 \le p\left(\vec{x}\right) < 1$, for all $\vec{x}$. Indeed, I can prove[1]:

Lemma 1. *. If $p : [0, \infty[^n \to [0, 1]$ is an innovation function that satisfies either (2) or (3), then we have that $0 \le p\left(\vec{x}\right) < 1$, for all $\vec{x}$ in $[0, \infty[^n$. If p is a continuous function that satisfies (1), then we have either $p\left(\vec{x}\right) \equiv 1$; or $0 \le p\left(\vec{x}\right) < 1$, for all $\vec{x}$. Finally, if $p\left(\vec{x}\right) \equiv 1$, p in an innovation function that satisfies (1).*

Since $p\left(\vec{x}\right) \equiv 1$ is a degenerate case of constant returns to scale, we will limit henceforth the discussion to the case where $0 \le p\left(\vec{x}\right) < 1$.

As a second step, let us define the function $g : [0, \infty[\to [0, 1[$ by the equation

$$g\left(y\right) = 1 - e^{-y}. \tag{4}$$

We can note g is a continuous and bijective mapping whose inverse is the function $h : [0, 1[\to [0, \infty[$, which is defined by the equation

$$h\left(z\right) = \ln\left(\frac{1}{1-z}\right). \tag{5}$$

We can note also g and h are strictly increasing on their respective domains of definition.

I can now expound the solution of Eqs. (1)–(3) and obtain a complete characterization of returns to scale, when the research technology is such that $0 \le p\left(\vec{x}\right) < 1$:

Proposition 1. *Let $p : [0, \infty[^n \to [0, 1[$ be a function. Then,*

(A) *There exists one and only one function $f : [0, \infty[^n \to [0, \infty[$ such that*

$$p\left(\vec{x}\right) = 1 - \exp\left(-f\left(\vec{x}\right)\right); \tag{6}$$

(B) *The function f is given by*

$$f\left(\vec{x}\right) = \ln\left(\frac{1}{1-p\left(\vec{x}\right)}\right); \tag{7}$$

(C) *p is an innovation function if, and only if, f is continuous and monotone increasing in all arguments; and p is strictly increasing in all arguments if, and only if, f is strictly increasing in all arguments;*

(D) *$p\left(\vec{0}\right) = 0$ if, and only if, $f\left(\vec{0}\right) = 0$;*

[1] All proofs are in the Appendix.