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Characterization of the core in full domain marriage problems

Duygu Nizamogullari[a,*,1], İpek Özkal-Sanver[b,c]

[a] Department of Mathematics, Piri Reis University, Turkey
[b] Department of Economics, İstanbul Bilgi University, Turkey
[c] Murat Sertel Center for Advanced Economic Studies, İstanbul Bilgi University, Turkey

HIGHLIGHTS

- We consider a marriage model where agents are allowed to stay self-matched.
- We analyse different definitions of converse consistency to this more general model.
- We characterize the core in this setting.

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ABSTRACT

In this paper, we study the core of two-sided, one-to-one matching problems. First, in a model in which agents have strict preferences over their potential mates and are allowed to remain single, we characterize the core as the unique solution that satisfies individual rationality, Pareto optimality, gender fairness, consistency, and converse consistency. Next, in a model that relaxes the constraint that agents have strict preferences over their potential mates, we show that no solution exists that satisfies Pareto optimality, anonymity, and converse consistency. In this full domain, we characterize the core by individual rationality, weak Pareto optimality, monotonicity, gender fairness, consistency, and converse consistency.

1. Introduction

The aim of this paper is to study consistency notions as well as fairness ideas applicable in two-sided, one-to-one matching problems. Consistency and converse consistency are two fundamental properties of solutions in variable population models, and several versions of the corresponding axioms have been studied in different economic problems.[2] Sasaki and Toda (1992) is the first paper that introduces consistency and converse consistency to the matching literature. Sasaki and Toda consider a restricted domain in which agents have strict preferences over their potential mates and are not allowed to remain single. In this restricted domain, these authors show that the core is the only natural (anonymous and efficient) solution that satisfies these two notions of consistency. A more recent paper by Klaus (2013) extends this characterization to the domain of "no odd rings" roommate problems.

Consistency is a kind of independence of irrelevant alternatives axiom (Thomson, 2009). For example, consider a Ph.D. program in which Ph.D. students who pass their preliminary exams are each assigned to a professor, and each professor can accept only one additional student per year. Suppose there are three students, a, b, and c, who have passed their preliminary exams and three professors, A, B, and C. Moreover, assume that each student has a strict preference over professors and that each professor has a strict preference over students. Then, suppose these agents are matched as follows: A is matched with a, B is matched with b, and C is matched with c. Then, before the public announcement of the results, suppose that Professor C is accepted by a university abroad as a visiting scholar and that student c transfers to that university. Then two professors, A and B, and two students, a and b remain. Applying the same matching process to these remaining agents, it is natural to

* Corresponding author. Tel.: +90 5322423776.
E-mail addresses: duygunizam@gmail.com, dnizamogullari@pru.edu.tr (D. Nizamogullari), isanver@bilgi.edu.tr (İ. Özkal-Sanver).

1 This manuscript is a part of her Ph.D. thesis at the Department of Economics, İstanbul Bilgi University.

2 For instance, bargaining problems (Lensberg, 1987, 1988), games in coalitional form with transferable utility (Hart and Mas-Colell, 1989; Moulin, 1985; Peleg, 1986, 1989, 1992; Sobolev, 1975), games in coalitional form without transferable utility (Peleg and Tijs, 1996; Tadenuma, 1992), bankruptcy and taxation (Chun, 1988; Young, 1987), general cost allocation problems (Moulin and Shenker, 1994), pricing mechanisms (McLean et al., 2004), fair division in economies with single-peaked preferences (Thomson, 1994), apportionment (Balinski and Young, 1982), two-sided matching (Klaus and Klijn, 2013; Özkal-Sanver, 2013; Sasaki and Toda, 1992; Toda, 2006), roommate problems (Can and Klaus, 2013; Klaus, 2013; Özkal-Sanver, 2010).

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expect that a is assigned to A and b is assigned to B. However, only consistent matching processes guarantee this property.

In contrast, converse consistency is a kind of decentralization axiom. For example, consider the scenario in which the matching process does not match A with a, B with b, and C with c. The Ph.D. program committee tests the accuracy of their decision in the following way. First, the committee considers professors A and B and students a and b. In this case, the matching process recommends assigning student a to professor A and student b to professor B. Next, the committee considers professors B and C and students b and c. In this case, the same matching process recommends assigning student b to professor B, and assigning student c to professor C. Finally, the committee considers professors A and C and students a and c. In this case, the same matching process recommends assigning student a to professor A and student c to professor C. Therefore, it is natural to expect that the application of the same matching process to the original problem will match A with a, B with b, and C with c. However, only conversely consistent matching processes guarantee this property. Thomson (2011) illustrated the concept of converse consistency by a jigsaw puzzle in which the correct positioning of each pair of pieces two-by-two guarantees correct positioning of all of the pieces in the puzzle.

Two-sided, one-to-one matching processes such as the assignment of Ph.D. students to professors are generally modeled under the nickname of "marriage problems". Marriage problems include two disjoint sets of agents, namely, men and women, who have preferences over the agents who belong to other set. In this paper, first we consider a model in which agents have strict preferences over their potential mates but are allowed to remain single. Individual rationality, which is a property of voluntary contribution, allows an agent to remain unmatched if the match to the mate is considered to be a worse outcome. Consistency is trivially adapted to the model, in which agents are allowed to remain single. However, there are several possible ways to adapt converse consistency to this model.[3] For example, Toda (1993) showed that for a model that allows agents to remain unmatched, there exists a solution other than the core that satisfies individual rationality, Pareto optimality, anonymity, consistency, and a very weak version of converse consistency. In this paper, we replace the weak version of converse consistency with a stronger version, which is also used by Klaus (2013), to show that the result characterized by Sasaki and Toda (1992) cannot be extended to a more general domain.

Under this more general framework, we characterize the core as the unique solution that satisfies individual rationality, Pareto optimality, gender fairness, consistency, and converse consistency. Gender fairness is a stronger version of anonymity.[4] If a solution is gender fair, then applying the solution after renaming men as women and women as men results in an outcome that has been permuted accordingly. Gender fairness is a combination of anonymity and external anonymity (when exchanging the roles of men and women).

Next, we consider a model in which we relax the condition that agents have strict preferences over their potential mates. If we allow indifferences, then the core fails to satisfy Pareto optimality. Furthermore, we show that no solution exists that satisfies Pareto optimality, anonymity, and converse consistency. We characterize the core as the unique solution that satisfies individual rationality, weak Pareto optimality, gender fairness, consistency, converse consistency, and a weak version of monotonicity. Another characterization of the core can be found in Toda (2006)[5] where the core of many-to-one matching problems is characterized by weak unanimity, consistency, Maskin monotonicity,[6] and population monotonicity.

The remainder of this paper is organized as follows. Section 2 presents basic notations and definitions, and Section 3 reports the results for the first model, in which agents have strict preferences over their potential mates but are allowed to remain single. Then, Section 4 reports the results for the second model, in which agents may have weak preferences over potential mates. Finally, Section 5 concludes the paper.

2. Notations and definitions

Let M and W be two disjoint universal sets, and let M be a non-empty and finite subset of M. Similarly, let W be a non-empty and finite subset of W. Then, a society is a union of some $M \subset \mathsf{M}$ and some $W \subset \mathsf{W}$. In the context of marriage, the set M stands for a set of men and the set W stands for a set of women.

Let $\mathsf{A} = \mathsf{M} \cup \mathsf{W}$ denote the universal set of agents, and let $\mathcal{A} = \{M \cup W\}_{M \subset \mathsf{M}, W \subset \mathsf{W}}$ be the set of all possible societies. For each society, $A = M \cup W \in \mathcal{A}$, and for each agent $i \in A$, the set of potential mates of agent i (including self-matches) is denoted by $A(i)$, which is defined as

$$A(i) \equiv \{i\} \cup \begin{cases} W & \text{if } i \in M \\ M & \text{if } i \in W. \end{cases}$$

Each agent $i \in A$ has a strict preference relation over $A(i)$ that is denoted by P_i. We will use tables throughout the paper to represent the preferences of the agents.[7] Let $\mathcal{P}$ denote the set of all possible preference profiles $P \equiv (P_i)_{i \in A}$.

A *matching* is a function $\mu : A \longrightarrow A$ such that $\mu(i) \in A(i)$ and $\mu^2(i) = i$ for all $i \in A$. Here, $\mu(i)$ is the mate of agent i under the matching μ. If $\mu(i) = i$, then agent i is self-matched or single. Let $\mathcal{M}(A)$ denote the set of all matchings for A.

A *(matching) problem* p is a pair $p = (A, P)$ in which A is a society and $P = (P_i)_{i \in A}$ is a profile of preferences over potential mates. Let $\mathbf{P}$ denote the set of all problems.

Let $p = (A, P) \in \mathbf{P}$ be an arbitrary problem. A matching $\mu \in \mathcal{M}(A)$ is *individually rational* for p if there is no agent $i \in A$ such that $iP_i\mu(i)$. Let $\mathcal{IR}(p)$ denote the set of all individually rational matchings. A pair of agents $(i, j) \in M \times W$ *blocks* a matching $\mu \in \mathcal{M}(A)$ if $jP_i\mu(i)$ and $iP_j\mu(j)$. A matching $\mu \in \mathcal{M}(A)$ is *stable* for p if it is individually rational for p and there is no pair $(i, j) \in M \times W$ that blocks μ at p. Let $\mathcal{S}(p)$ denote the set of all stable matchings.

Given a problem $p = (A, P) \in \mathbf{P}$ and two matchings $\mu, \mu' \in \mathcal{M}(A)$ with $\mu' \neq \mu$, μ *Pareto dominates* μ' if for all $i \in A$, $\mu(i)P_i\mu'(i)$ whenever $\mu(i) \neq \mu'(i)$. A matching $\mu \in \mathcal{M}(A)$ is *Pareto optimal* for p if there exists no matching $\mu' \in \mathcal{M}(A)$ that Pareto dominates μ. Let $\mathcal{PO}(p)$ denote the set of all Pareto optimal matchings for p.

A *solution* φ is a correspondence that assigns a non-empty subset $\varphi(p) \subseteq \mathcal{M}(A)$ to each p. The *core* is the solution that assigns to each problem p its set of stable matchings.

Given a problem $p = (A, P)$ and a subset of agents $N \subseteq A$, the problem $p' = (N, P_{|N}) \in \mathbf{P}$ is the *reduced problem* that is related to $p = (A, P)$, where $P_{|N}$ is the restricted preference profile of the agents in N. Similarly, given a matching $\mu \in \mathcal{M}(A)$ and a subset of agents $N \subseteq A$, $\mu_{|N} \in \mathcal{M}\left(N \underset{i \in N}{\cup} \{\mu(i)\}\right)$ is a reduced matching of μ

[3] One can refer to the Concluding Remarks for the definitions of these three versions of converse consistency and a brief discussion of them.

[4] See Özkal-Sanver (2004).

[5] The characterization result of Toda (2006) has straightforward adaptation to marriage problems.

[6] For a detailed analysis of Maskin monotonicity in matching problems one can refer to the paper by Kara and Sönmez (1996), which shows that any Pareto optimal, individually rational, and Maskin monotonic solution should be a supersolution of the core.

[7] The column $\begin{matrix} P_i \\ j \\ k \\ l \end{matrix}$ represents jP_ikP_il.