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Optimal portfolio with vector expected utility

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HIGHLIGHTS

- We propose a generalisation to ambiguity of the mean–variance analysis.
- Founded on the Vector Expected Utility's certainty equivalent's Taylor expansion.
- We solve for optimal portfolios when returns are ambiguous.
- We test the robustness to axiomatic specifications of existing results.
- We propose a novel analysis of the "home-bias puzzle" with two ambiguous assets.

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ABSTRACT

We study the optimal portfolio selected by an investor who conforms to Siniscalchi (2009)'s Vector Expected Utility's (VEU) axioms and who is ambiguity averse. To this end, we derive a mean–variance preference generalised to ambiguity from the second-order Taylor–Young expansion of the VEU certainty equivalent. We apply this Mean–Variance Variability preference to the static two-assets portfolio problem and deduce asset allocation results which extend the mean–variance analysis to ambiguity in the VEU framework. Our criterion has attractive features: it is axiomatically well-founded and analytically tractable, it is therefore well suited for applications to asset pricing as proved by a novel analysis of the home-bias puzzle with two ambiguous assets.

1. Introduction

Since the seminal works of Markowitz (1952) and Tobin (1958), mean–variance preferences have been the cornerstone of optimal portfolio theory. An investor with mean–variance preference having to select risky assets will rank uncertain portfolio returns r according to the following evaluation of their utility:

$$u_{\mathrm{MV}}(r) = \mathbf{E}_{\mathbf{p}}(r) - \frac{\gamma}{2}\mathbf{var}_{\mathbf{p}}(r)$$

where $\mathbf{p}$ is a given probability and γ is a measure of the aversion to variance. The foundations of the mean–variance preferences and the link between risk and variance for "small risks" are to be found in the classical Arrow–Pratt (Pratt, 1964; Arrow, 1965) approximation of the Expected Utility (EU) certainty equivalent: for an investor with certain wealth w considering a risky investment h, the Taylor expansion to the second order of its certainty equivalent is given by:

$$c(w+h) = w + \mathbf{E}_{\mathbf{p}}(h) - \frac{1}{2}\gamma(w)\mathbf{var}_{\mathbf{p}}(h) + o(\mathbf{var}_{\mathbf{p}}(h))$$

where $\gamma(w) = -u''(w)/u'(w)$ is the absolute risk aversion coefficient of the *Bernoulli utility function*[1] u.

While the mean–variance analysis remains the workhorse of modern portfolio theory, it is well known that empirical data cannot be fully rationalised in this context, especially, the equity premium cannot be explained by a risk premium only (the "equity premium puzzle", Mehra and Prescott, 1985) and international portfolios are under diversified (the "home-bias puzzle", French and Poterba, 1991). A large amount of literature has endeavoured to explain these "puzzles", analysing different shortcomings of the classical paradigm. Among these, recent advances in decision theory aimed at generalising the EU framework have allowed to study the effect on asset prices of *ambiguity*: situations where the information available to the investor is too imprecise to be summarised by a unique probability distribution over events.[2] This paper fits into this field of research: its main contribution is to propose a mean–variance preference generalised to ambiguity

[1] In Mas-Colell et al. (1995)'s terminology, see note 12 p. 184.

[2] Recent general references on the subject of ambiguity include Wakker (2008); Etner et al. (2009) and Gilboa and Marinacci (2011).

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using Siniscalchi (2009)'s Vector Expected Utility (VEU). We study the conditions for existence and calculate the second order Taylor–Young expansion of the VEU certainty equivalent from which we derive a *Mean–Variance Variability* preference. This flexible and tractable criterion allows not only to retrieve the existing results for an optimal portfolio with one risky and one ambiguous asset but also to show new results with two ambiguous assets, which we apply to the discussion of the home-bias puzzle.

Several non EU decision theoretic models have been successfully applied to the field of finance and to the discussions of the "puzzles". Among these applications, some have sought to improve the mean–variance preferences: especially Maccheroni et al. (2013) derive a mean–variance model adjusted for ambiguity from a quadratic approximation of the certainty equivalent of the smooth model of decision making under ambiguity (Klibanoff et al., 2005, henceforth KMM). Our work is closely related to this paper which provided the impetus for our research, but is set in a different axiomatic framework hence uses different mathematical tools. While the KMM model has been successfully applied to numerous asset markets problems, we have chosen the VEU model for its specific axiomatic foundations which make it very well suited to financial applications: its central axiom of complementary independence, which will be detailed below, has a very clear and intuitive behavioural interpretation in a portfolio application. Finally the new results which have been obtained in this paper with two ambiguous assets justify by themselves the choice of the VEU model.

A decision maker (DM) conforms to the VEU set of axioms if and only if (iff) she ranks uncertain prospects f, functions from a state space S to a set X of consequences which is a mixture space, via the functional:

$$V(f) = \mathbf{E_p}(\tilde{u} \circ f) + A(\{\mathbf{E_p}(\zeta_i \cdot \tilde{u} \circ f)\}_{0 \leqslant i < n}) \quad (1)$$

where $\tilde{u}: X \to \mathbb{R}$ is a von Neumann–Morgenstern utility function and $A: \mathbb{R}^n \to \mathbb{R}$ is a function such that for any $\varphi \in \mathbb{R}^n$, $A(-\varphi) = A(\varphi)$ and $A(0_n) = 0$. There are two parts to this evaluation: the expected utility of the act according to a baseline probability $\mathbf{p}$ and an adjustment to this baseline evaluation, function of the *variability* of the utility profile and of the DM's attitude towards ambiguity.

The baseline probability $\mathbf{p}$ is a key feature of the VEU model as it is revealed by the preferences of the DM over *complementary acts*. These are acts f and $\bar{f}$ such that for any states s and $s' \in S$, $\frac{1}{2}f(s) + \frac{1}{2}\bar{f}(s) \sim \frac{1}{2}f(s') + \frac{1}{2}\bar{f}(s')$ which implies that their utility profiles sum to a constant: $\tilde{u} \circ f = k - \tilde{u} \circ \bar{f}$ for some $k \in \mathbb{R}$. In the words of Siniscalchi, they are "the preference counterpart of algebraic negation". In a portfolio application, assuming linear utility, a long and a short position of the same value in the same asset are straightforward examples of complementary acts. The central insight of the VEU model is that complementary acts have the same utility variability, *i.e.* the same ambiguity, hence have to be ranked according to their baseline expected utility only. Therefore "preferences over complementary acts uniquely identify the baseline prior". As an illustration, consider (Ellsberg, 1961)'s three colour single urn experiment: a ball is drawn from an urn containing 30 red balls and 60 black and yellow balls with the proportion of black and yellow balls unknown. Assuming linear utility, the act $(10, R; -10, B; 0, Y)$ that yields \$10 if a red ball is drawn and costs \$10 if a black ball is drawn and the act $(-10, R; 10, B; 0, Y)$ are complementary: they embed the same ambiguity and if the DM is indifferent between these two, we can derive that $\mathbf{p}(R) = \mathbf{p}(B)$. If the DM is also indifferent between $(10, R; 0, B; -10, Y)$ and $(-10, R; 0, B; 10, Y)$, we can infer that she is using the uniform prior as her baseline probability over the state space $S = \{R, B, Y\}$.

The adjustment to the baseline evaluation refers to the notion of *crisp* acts, which has been introduced by Ghirardato et al. (2004, henceforth GMM). They characterise the *unambiguous preference* as the maximal[3] restriction satisfying the independence axiom of the complete DM preference. This preference is incomplete and has a Bewley (2002) representation by a unanimity criterion over a set of priors $\mathcal{C}$. Acts which have the same expected utility for all the priors in $\mathcal{C}$ are crisp, hence non crisp acts have "variable utility profiles". Using the Hilbert space structure of $\mathsf{L}^2(\mathbf{p})$, Siniscalchi proves that the subspace of crisp acts $\mathtt{C}$ and the subspace of non crisp acts $\mathtt{NC}$ are orthogonal complements, hence that any act can be decomposed into a crisp component and a "purely ambiguous" one. With $\{\zeta_i\}_{0 \leqslant i < n}$ a basis for the subspace $\mathtt{NC}$, the purely ambiguous component can be written as a linear combination of the vectors ζ_i which are named *adjustment factors* and are interpreted as sources of ambiguity. To give more intuition on this construction, we can elaborate on Siniscalchi's suggestion to draw a parallel with factor pricing models in finance: in these models, "expected asset returns are determined by a linear combination of their covariances with variables representing the risk factors" (Ferson, 2003). Cochrane (2005) proposes as risk factors explaining the asset returns: returns on broad based portfolios, interest rates, growth in GDP, investment but also the term premium or the dividend/price ratios. If the DM entertains more than one possible probabilistic scenario for these risk factors, then they can also be seen as the drivers of ambiguity. In this paper we will use a *sharp* VEU representation, that is one where the basis $\{\zeta_i\}_{0 \leqslant i < n}$ is made orthonormal by an application of the Gram–Schmidt procedure. The sources of ambiguity are then $\mathbf{p}$-independent, and exposure to one source cannot hedge the ambiguity coming from another. But the interpretation of independent sources of ambiguity as macroeconomic variables is more difficult, a difficulty that also arises with factor pricing models, for example when orthogonalised risk factors are used in the Arbitrage Pricing Theory.[4]

Going back to Eq. (1), it can now be seen that the argument of the adjustment function A is the vector of coordinates of the utility profile in $\mathtt{NC}$, which can be read as the correlations of the utility profile with each source of ambiguity. Thanks to this construction, the VEU evaluation nicely reduces to EU for crisp acts and reflects complementarities among ambiguous acts. This can again be illustrated using Ellsberg's three colour urn, as in the original paper: let ζ_0 be the random variable such that $\zeta_0(R) = 0$, $\zeta_0(B) = 1$ and $\zeta_0(Y) = -1$ and let $A(\varphi) = -|\varphi|$. Assuming the uniform prior that we derived above, any act f is evaluated through: $V(f) = \frac{1}{3}(f(R)+f(B)+f(Y)) - |\frac{1}{3}(f(B)-f(Y))|$. One can check that this evaluation is consistent with the preferences reported in Ellsberg (1961): $V[(10, R; 0, B; 0, Y)] > V[(0, R; 10, B; 0, Y)]$ but $V[(10, R; 0, B; 10, Y)] < V[(0, R; 10, B; 10, Y)]$ highlighting the complementarity of the payoffs on the events B and Y in the last act.

The mathematical details of this setup are exposed in Section 2. In Section 3 we give some sufficient conditions for the VEU certainty equivalent to be differentiable and we compute the second-order Taylor–Young expansion for a "small" incremental act when the DM is *ambiguity averse* in the sense of Schmeidler (1989), *i.e.* a DM with a weak preference for mixtures. The ambiguity averse case highlights the central role of the Hessian of the adjustment function A: the $\mathbb{R}^n$ rotation matrix made of its eigenvectors induces a rotated basis of the subspace of non-crisp acts $\mathtt{NC}$ made of normalised sources of ambiguity which are still $\mathbf{p}$-independent but are also not correlated for the DM's tastes, in the sense that there are no cross terms in her evaluation of an act spanning several sources. Section 4 delves further into the properties of the quadratic approximation and proves the link between the DM aversion to these sources of ambiguity and the

3 In the sense of set inclusion.

4 See also the discussion in Ferson (2003, Section 2.7).