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Stationary distribution of a stochastic SIQR epidemic model with saturated incidence and degenerate diffusion

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HIGHLIGHTS

- A regime-switching stochastic SIRS epidemic model with nonmonotone incidence rate is studied.
- We establish sufficient conditions for the diseases to die out with probability one.
- The Markov semigroup theory be employed to obtain the existence of a unique stable stationary distribution.

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ABSTRACT

The objective of this paper is to study stationary distribution of a stochastic SIQR epidemic model with saturated incidence and degenerate diffusion. Since the diffusion matrix is degenerate, the uniform ellipticity condition is not satisfied. The Markov semigroup theory will be used to obtain the existence of a unique stable stationary distribution. A threshold dynamic determined by the basic reproduction number R_0^s is established: the disease can be eradicated almost surely if $R_0^s < 1$, whereas if $R_0^s > 1$, it has an endemic stationary distribution which leads to the stochastic persistence of the disease.

1. Introduction

Mathematical models have become important tools in better understanding of epidemiological patterns and disease control for a long time, since the pioneer work of Kermack and McKendrick [1], which reveals the underlying mechanisms that influence the transmission and control of infectious diseases. It is now believed that mathematical models have been important tools in analyzing the spread and control of infectious diseases, qualitatively and quantitatively [2–4]. Over the centuries quarantine was considered as the most effective method in controlling the spread of disease and as soon as possible eliminate epidemic disease. It has been used to curb disease such as leprosy, plague, cholera, typhus, yellow fever, smallpox, diphtheria, tuberculosis, measles, mumps, ebola, lassa fever, rinderpest, foot and mouth, psittacosis, Newcastle disease and rabies [5]. In an effort to understand the effect of quarantine on endemic infectious diseases, Joshi et al. [6] established the

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following SIQR epidemic model which incorporates saturated incidence rate with quarantine:

$$\begin{cases} dS(t) = \left(A - \frac{\beta S(t)I(t)}{1+\alpha S(t)} - \mu S(t)\right)dt, \\ dI(t) = \left(\frac{\beta S(t)I(t)}{1+\alpha S(t)} - (\gamma + \delta + \mu + \mu_1)I(t)\right)dt, \\ dQ(t) = \left(\delta I(t) - (\theta + \mu + \mu_2)Q(t)\right)dt, \\ dR(t) = \left(\gamma I(t) + \theta Q(t) - \mu R(t)\right)dt, \end{cases} \tag{1.1}$$

where $S(t), I(t), Q(t), R(t)$ are the numbers of susceptible, infectious, quarantined and recovered individuals at time t, respectively. The parameters have the following biological meanings: A is the recruitment rate of the population, $\frac{\beta S(t)I(t)}{1+\alpha S(t)}$ is saturation incidence rate, β is the average number of adequate contacts of a person per unit time, α measures the appropriate preventions taken by susceptible for epidemic control. It is easy to obtain $\frac{\beta SI}{1+\alpha S} \to \frac{\beta}{\alpha}I$ as $S \to \infty$. μ is the natural death rate of the population, μ_1, μ_2 are the extra disease-related death rate constant in compartments I and Q respectively. γ, θ denote the recover rates from group I, Q to R respectively, δ denotes the removal rate from I. Their study showed that the system (1.1) has a disease-free equilibrium $E_0 = (\frac{A}{\mu}, 0, 0, 0)$, the basic reproduction number for system (1.1) can be defined as $R_0 = \frac{A\beta}{(A\alpha+\mu)(\delta+\gamma+\mu+\mu_1)}$. Theoretical results show that if $R_0 < 1$, the disease-free equilibrium E_0 is globally asymptotically stable in invariant set $\Gamma = \{(S, I, Q, R) \in \mathbb{R}^4_+ : \frac{A}{\mu+\mu_1+\mu_2} < S + I + Q + R < \frac{A}{\mu}\}$, whereas if $R_0 > 1$, then E_0 is unstable and there has a unique endemic equilibrium $E^* = (S^*, I^*, Q^*, R^*)$ which is globally asymptotically stable.

In fact, the nature of epidemic growth and spread are usually influenced by environmental variations. For example, the disease transmission rate β in epidemic models was modified for meteorological factors because survivals and infectivity of many viruses and bacteria are better in damp conditions with little ultraviolet light [7]. Therefore, based on the biological system subject to stochastic environmental conditions, stochastic models could be more appropriate way of modeling epidemics under various circumstances. It also has been shown that some stochastic epidemic models can provide an additional degree of realism in comparison with their deterministic system. Recently there has a great interest in stochastic epidemic models [7–25]. Wang et al. [8] investigated a classical SIRS epidemic model with the infectious forces under intervention strategies. They proved that the reproduction number $\hat{R}_0^s$ can be used to govern the stochastic dynamics of SDE model. And theoretical results show that if $\hat{R}_0^s < 1$, under mild extra conditions, the SDE system has a disease-free absorbing set which means the extinction of disease with probability one, whereas if $\hat{R}_0^s > 1$, under mild extra conditions, it has an endemic stationary distribution which leads to the stochastical persistence of the disease. In particular, Jiang et al. [23] formulates and analyzes a stochastic SIS epidemic model with perturbed disease transmission coefficient. They deduced a threshold dynamic determined by the basic reproduction number $\tilde{R}_0^s$, and shown analytically that if $\tilde{R}_0^s < 1$, the disease dies out in probability; whereas if $\tilde{R}_0^s > 1$, the densities of the distributions of the solution can converge in L^1 to an invariant density. Thanks to the insightful works of Wang [8] and Jiang [23], we have one of interesting findings is that random fluctuations can suppress disease outbreak, which can provide us some useful control strategies to regulate disease dynamics. Therefore, we intend to consider the SIQR epidemic model (1.1) incorporates the influence of the random environment, moreover, we take into account the effect of randomly fluctuating environment into model (1.1) by assuming $\beta \to \beta + \sigma dB(t)$, then we can obtained the SDE model as follows

$$\begin{cases} dS(t) = \left(A - \frac{\beta S(t)I(t)}{1+\alpha S(t)} - \mu S(t)\right)dt - \frac{\sigma S(t)I(t)}{1+\alpha S(t)}dB(t), \\ dI(t) = \left(\frac{\beta S(t)I(t)}{1+\alpha S(t)} - (\gamma + \delta + \mu + \mu_1)I(t)\right)dt + \frac{\sigma S(t)I(t)}{1+\alpha S(t)}dB(t), \\ dQ(t) = \left(\delta I(t) - (\theta + \mu + \mu_2)Q(t)\right)dt, \\ dR(t) = \left(\gamma I(t) + \theta Q(t) - \mu R(t)\right)dt, \end{cases} \tag{1.2}$$

where $B(t)$ is a real-valued standard Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all P-null sets); σ^2 represent the intensities of the white noise. And the state space of SDE model (1.2) is $\mathbb{X}$.

Since stationary distribution of stochastic system has the same status as the equilibrium point of deterministic system, many authors considered the existence of a unique ergodic stationary distribution (e.g. [9,11,13,15,18,20–22]). Notice that in those literatures the proofs of main results depend heavily on the uniform ellipticity condition: the diffusion matrix $\left(a_{ij}(x)\right)$ satisfies $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq M|\xi|^2$ for all $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ with some constant $M > 0$ for all $x \in \mathbb{R}^n$. However, the diffusion matrix Λ of model (1.2) in this paper is not satisfy the uniform ellipticity condition, where

$$\Lambda = \begin{pmatrix} \frac{\sigma^2 S^2 I^2}{(1+\alpha S)^2} & \frac{-\sigma^2 S^2 I^2}{(1+\alpha S)^2} & 0 & 0 \\ \frac{-\sigma^2 S^2 I^2}{(1+\alpha S)^2} & \frac{\sigma^2 S^2 I^2}{(1+\alpha S)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thanks to the insight work of Rudnicki [26], in this paper, we adopt a different tool which comes from the Markov semigroup theory to obtain a stable stationary distribution.