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Stochastic sandpile model on small-world networks: Scaling and crossover

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HIGHLIGHTS

- A dissipative stochastic sandpile model is studied on one and two dimensional small-world networks.
- The effect of role of dimensionality of the underlying regular lattice and evolution of regular lattice to the small-world on critical behaviour of sandpile model is analysed.
- Several new geometrical quantities of avalanches such as toppling surface and its fragmentation, compactness and fluctuation in the fragment size are characterized.
- Various regimes of avalanche size which are separated by crossover sizes are identified and their scaling relation are developed.
- Coexistence scaling form of the distributions and the expectation value are developed and numerically verified.
- A generalized scaling form of sand transport behaviour is developed which explains the crossover behaviour from diffusive to super-diffusive nature when small-world network evolves from regular lattice to random network.

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ABSTRACT

A dissipative stochastic sandpile model is constructed on one and two dimensional small-world networks with shortcut density ϕ, $\phi = 0$ represents a regular lattice whereas $\phi = 1$ represents a random network. The effect of the transformation of the regular lattice to a small-world network on the critical behaviour of the model as well as the role of dimensionality of the underlying regular lattice are explored studying different geometrical properties of the avalanches as a function of avalanche size s in the small-world regime ($2^{-12} \le \phi \le 0.1$). For both the dimensions, three regions of s, separated by two crossover sizes s_1 and s_2 ($s_1 < s_2$), are identified analysing the scaling behaviour of average height and area of the toppling surface associated with an avalanche. It is found that avalanches of size $s < s_1$ are compact and follow Manna scaling on the regular lattice whereas the avalanches with size $s > s_1$ are sparse as they are on network and follow mean-field scaling. Coexistence of different scaling forms in the small-world regime leads to violation of usual finite-size scaling which were valid on the regular lattice as well as on the random network independently. Simultaneous appearance of multiple scaling forms are characterized by developing a coexistence scaling theory. As SWN evolves from regular lattice to random network, a crossover from diffusive to super-diffusive nature of sand transport is observed and scaling forms of such crossover is developed and verified.

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1. Introduction

The spontaneous emergence of spatio-temporal correlation and hence the criticality in many natural phenomena are found to be outcome of self-organized criticality (SOC) [1–3]. The appearance of SOC on real-world complex networks is ubiquitous in nature starting from Biology to Astrophysics, technology to society. Examples are: avalanche mode of activity in the neural network of brain [4,5], earthquake dynamics on the network of faults in the crust of the earth [6], rapid rearrangement of the magnetic field network in the corona [7], propagation of information through a network with a malfunctioning router causing the breakdown of the Internet network [8], breakdown of the electric power grid [9], knowledge creation in coevolving social network [10] and many others.

On the other hand, small-world network (SWN) [11] has an interesting property. It not only interpolates between the regular lattice (RL) and the random network (RN) through the emergence of inherent length scale, but also it preserves the properties of both RL and RN, namely high "clustering-coefficient" (concept of neighbourhood) and "small-world-effect" (small average shortest distance between any two nodes) respectively [12–14]. It has already been observed that introduction of internal length scale in the system has a significant influence on the critical behaviour of self-organization processes [15]. It is then intriguing to study the models of SOC on SWN as it also possesses the internal length scale.

Sandpile, a prototypical deterministic model to study SOC, introduced by Bak, Tang, and Wiesenfeld (BTW) [16,17] on RL gives rise to anomalous (multi) scaling [18–20]. A mean-field (MF) solution of the avalanche dynamics was obtained applying theory of branching processes [21–23]. Interestingly, the deterministic BTW model found to exhibit MF scaling [24–26] when studied on RN rather than the anomalous scaling on RL. A crossover from the anomalous scaling on RL to MF scaling on RN with coexistence of the both the scaling in the SWN regime was found to occur in the study of BTW model on SWN [26]. However, a stochastic sandpile model (SSM) which incorporates random distribution of sand grains during avalanche, exhibits a scaling behaviour with definite critical exponents that follows finite-size scaling (FSS) when studied on RL. SSM defines a robust universality class called Manna class [27,28]. Not only SSM shows robust scaling behaviour than the deterministic BTW model, it is also able to explain certain experimentally observed avalanche behaviour [29]. Stochasticity in the model can also be incorporated in different ways, e.g. occurrence of toppling in a probabilistic manner [30], stochasticity in the update scheme [31]. By virtue of symmetry, all those stochastic models fall under Manna universality class. The other symmetries in the toppling rule such as deterministic, directed or rotational symmetry lead the sandpile models to their respective universality classes [32,33]. Interestingly, SSM and its variants (like stochastic parallel Zhang (SPZ) model [31]) are found to be the most studied models in various dimensions in SOC literature [34–36]. Incorporating synaptic noise the SPZ model was specially studied in the context of neuronal avalanches on the Newman–Watts small-world networks. Such studies not only report the multiple scaling forms, stochastic and MF, but also provide understanding of structural and dynamical origin of the MF result in sandpile models [37,38]. It has been argued that the noisy local dynamical rule can also increase the effective range of interaction that lead to MF result. In a recent article of Bhaumik and Santra [39], the scaling of various types of avalanches, dissipative and non-dissipative, of SSM on SWN were reported and compared with those on the RL [40]. However, a detailed study of the avalanche properties of SSM on an evolving SWN, from RL to RN, and the effect of dimensionality on such critical behaviour are not reported so far to the best of our knowledge.

In this paper, a dissipative stochastic sandpile model (DSSM) in which dissipation of the sand occurs in the bulk of the system in an annealed manner rather than at the boundary is developed and studied on SWN as a function of shortcut density ϕ in one and two dimensions at specific dissipation rates. The effect of the transformation of the RL to a SWN and finally to the RN on the critical behaviour of the model as well as the role of dimensionality of the underlying RL are explored studying different geometrical properties of the avalanches as a function of avalanche size. Beside coexistence of both the scaling forms of RL and RN in the SWN regime, various scaling forms of different avalanche properties are identified and numerically verified. A qualitative explanation has been provided for different scaling forms in terms of diffusivity of sand grains based on numerical evidences.

2. The model: DSSM on SWN

Following Newman–Watts model [13], SWN is generated by adding shortcuts between two randomly chosen sites both of a one dimensional (1d) lattice and those of a two dimensional (2d) square lattice. The shortcut density, number of added shortcuts per existing bond, is defined as $\phi = N_\phi/(dL^d)$ where N_ϕ is the number of shortcuts added and dL^d is the number of bonds (without shortcuts) present in a d-dimensional lattice of linear size L with periodic boundary conditions in both the directions. Care has been taken to avoid self-edges of any node and multi-edges between any two nodes. The average degree of the network changes with ϕ as $\langle k\rangle = 2(1+\phi)$ in 1d and $\langle k\rangle = 4(1+\phi)$ in 2d. The system corresponds to a RL as $\phi \to 0$, whereas in the limit $\phi \to 1$ it corresponds to a RN which is characterized by Poisson degree distribution [12]. For the intermediate values of ϕ such as 2^{-12}–2^{-3}, the system behaves like SWN characterized by high clustering coefficient and small average path between any two nodes [12,13,26,41]. To study the sandpile dynamics on an SWN, a suitable value of ϕ is chosen and the SWN is driven by adding sand grains, one at a time, to randomly chosen nodes. If the height h_i of the sand column at the ith node becomes greater than or equal to the predefined threshold value h_c, which is equal to 2 here, the ith node will topple and the height of the sand column of the ith node will be reduced by h_c. The toppled two sand grains are then distributed among two of its randomly selected adjacent nodes which are connected to the toppled node either by